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Note on compatible operations of modular lattices

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In [2], G. Grätzer proved that every compatible operation of a bounded distributive lattice L is order-preserving (even a polynomial function) if and only if L does not contain a non-trivial boolean interval. D. Dorninger and G. Eigenthaler [1] noticed that this equivalence holds for distributive lattices in general. The aim of this note is to prove this equivalence for modular lattices whose intervals have finite primitive length.

Let us recall that an operation of a lattice is defined to be *compatible* if it preserves all congruence relations of the lattice. A modular lattice L has *finite primitive length* (cf. [5]) if and only if there is a natural number n such that $k \leq n$ for every sequence $[c_1, d_1], \ldots, [c_k, d_k]$ of projective intervals of L (i.e. $[c_1, d_i] \approx [c_j, d_j]$ for $1 \leq i, j \leq k$) with $d_i \leq c_{i+1}$ for $i = 1, \ldots, k-1$; L is distributive if and only if one can choose $n = 1$. A unary polynomial function is also called a *translation*. For the proof of the main result we need the following lemma which is a direct consequence of Satz 13.3 and Lemma 19.2 in [3].

LEMMA 1. *Let L be a modular lattice and let $a, b, a', b' \in L$ with $a < b$ and $a' < b'$. If (a', b') is contained in the congruence relation $\theta(a, b)$ generated by (a, b) then there exist elements c and d and a translation t of L such that $a \leq c < d \leq b$, $a' = t(c) < t(d) \leq b'$, and $[c, d] \approx [t(c), t(d)]$; furthermore, there is a translation t^* with $t^*(t(c)) = c$ and $t^*(t(d)) = d$.*

THEOREM 2. *For a modular lattice L whose intervals have finite primitive length the following conditions are equivalent:*

(i) *Every compatible operation of L is order-preserving.*

(ii) *L does not contain a non-trivial boolean interval.*

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Proof. (i) $\Rightarrow$ (ii): Suppose that L contains a boolean interval $[a, b]$ with $a \neq b$. Let y' be the complement of y in $[a, b]$. Then $f: L \to L$ defined by $f(x) := ((x \vee a) \wedge b)'$ is compatible but not order-preserving because of $f(a) = b$ and $f(b) = a$ (cf. [1], Corollary 1 and 2 of Theorem 3).

(ii) $\Rightarrow$ (i): Suppose there is an n-ary compatible operation p of L such that $p(a_1, \ldots, a_n) \not\leq p(b_1, \ldots, b_n)$ for some elements $a_i \leq b_i$ $(i = 1, 2, \ldots, n)$ of L. Then there exists an i with $p(a_1, \ldots, a_i, b_{i+1}, \ldots, b_n) \not\leq p(a_1, \ldots, a_{i-1}, b_i, \ldots, b_n)$; hence we already have a unary compatible operation $f(x) := p(a_1, \ldots, a_{i-1}, x, b_{i+1}, \ldots, b_n)$ of L such that $f(a_i) \not\leq f(b_i)$ of the elements $a_i < b_i$. Since $(f(b_i), f(a_i) \vee f(b_i))$ is contained in the congruence relation $\theta(a_i, b_i)$ generated by (a_i, b_i), by Lemma 1, there exist elements c and d and a translation t of L with $a_i \leq c < d \leq b_i$ and $f(b_i) = t(c) < t(d) \leq f(a_i) \vee f(b_i)$. Define $g(x) := t^*((f((x \vee a_i) \wedge b_i) \vee t(c)) \wedge t(d))$ for all $x \in L$; of course, g is a compatible operation of L and $g(a_i) = d$, $g(b_i) = c$. The finite primitive length of $[c, d]$ guarantees that there is a maximal number of elements $a_i \leq c_1 < d_1 \leq c_2 < d_2 \leq \cdots \leq c_k < d_k \leq b_i$ with $[c_j, d_j] \approx [c_{j+1}, d_{j+1}]$ for $j = 1, 2, \ldots, k-1$ and $c \leq c_m < d_m \leq d$ for some m. The intervals $[c_j, d_j]$ have to be distributive, otherwise each would contain non-trivial intervals $[c_{j1}, d_{j1}] \approx [c_{j2}, d_{j2}]$ with $d_{j1} < c_{j2}$ in contradiction to the maximality of k. Let u be a translation from $[c_m, d_m]$ onto $[c_1, d_1]$ and let v be a translation from $[c_1, d_1]$ onto $[c_k, d_k]$; furthermore, let $g_1(x) := u((g(x) \vee c_m) \wedge d_m)$ and $g_k(x) := v(g_1(x))$ for all $x \in L$. Suppose $g_1(c_1) \neq d_1 = g_1(a_i)$; again by Lemma 1, there would be non-trivial projective subintervals of $[a_i\, c_1]$ and $[g_1(c_1), g_1(a_i)]$ in contradiction to the maximality of k. Thus, $g_1(c_1) = d_1$ and hence $g_k(c_k) = d_k$. A dual argument yields $g_k(d_k) = c_k$. Suppose that there is an element x in $[c_k, d_k]$ with $x \vee g_k(x) \neq d_k = g(c_k)$; again by Lemma 1, there would be non-trivial projective subintervals of $[c_k, x]$ and $[x \vee g_k(x), d_k]$ in contradiction to the maximality of k. Thus, $x \vee g_k(x) = d_k$ and dually $x \wedge g_k(x) = c_k$ for all $x \in [c_k, d_k]$. Since $[c_k, d_k]$ is distributive, $[c_k, d_k]$ is boolean.

COROLLARY 3. *Let M be a modular lattice of finite primitive length in which every non-trivial interval contains a covering pair of elements and let P be a partially ordered set for which $\mathbf{2}^P$, i.e. the lattice of all order-preserving maps from P into the two-element lattice $\mathbf{2}$, does not contain a non-trivial boolean interval. Then every compatible operation of M^P, i.e. the lattice of all order-preserving maps from P into M, is order-preserving.*

Proof. Let $[f, g]$ be a non-trivial interval of M^P. Since $f < g$, there exists $y \in P$ with $f(y) < g(y)$. By assumption, there is covering pair $a \prec b$ in $[f(y), g(y)]$. Let $\bar{a}$ and $\bar{b}$ be the constant maps from P into M with $\bar{a}(x) = a$ and $\bar{b}(x) = b$ for all $x \in P$. Define $f_1 := (f \vee \bar{a}) \wedge \bar{b}$ and $g_1 := (g \vee \bar{a}) \wedge \bar{b}$. Then $f_1(y) = (f(y) \vee a) \wedge b = a$

and similarly $g_1(y) = b$. Thus, $\bar{a} \leq f_1 < g_1 \leq \bar{b}$. Now, $[\bar{a}, \bar{b}] \cong \mathbf{2}^P$ wherefore $[f_1, g_1]$ and hence $[f, g]$ cannot be boolean. By the theorem, every compatible operation of M^P is order-preserving (M^P has finite primitive length because it is a subdirect power of M).

Remark 1. The corollary can be generalized to all subdirect powers of M containing all constant functions and having the property that for every covering pair $a \prec b$ of M the interval $[a, \bar{b}]$ of the subdirect power does not contain a boolean interval.

Remark 2. In the theorem the assumption of finite primitive length is indispensable because there are infinite simple modular lattices not containing a non-trivial boolean interval (s. [4]) and in such a lattice every operation is compatible, but not always order-preserving

$$\left(\text{e.g. for} \quad a < b \quad \text{define} \quad f(x) := \begin{matrix} a & \text{if} & x \neq a \\ b & \text{if} & x = a \end{matrix}\right).$$

PROBLEM. Which modular lattices of finite primitive length have the property that every compatible operation is already a polynomial function?

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