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REMARK ON COMPATIBLE AND ORDER-PRESERVING FUNCTION ON LATTICES

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1. Introduction

Let k be a positive integer. If L is a lattice then $F_k(L)$ denotes the lattice of all functions f: $L^k \to L$. A function $f \in F_k(L)$ is called *compatible* if for any congruence relation Θ of L $a_i \equiv b_i(\Theta)$, $i=1, 2, \dots, k$ imply $f(a_1, \dots, a_k) = f(b_1, \dots, b_h)(\Theta)$, f is called *order-preserving*, if $a_i \leqq b_i$, $i=1, \dots, k$, implies $f(a_1, \dots, a_k) \leqq f(b_1, \dots, b_k)$. The set of all k-place compatible functions on L denoted by $C_k(L)$, the set of all k-place order-preserving functions on L is $OF_k(L)$. In this paper we deal with the following problem: which modular lattices satisfy $C_k(L) \subset OF_k(L)$?

For distributive lattices this problem was solved by D. DORNINGER and G. EIGENTHALER [1]:

THEOREM 1. *Let L be a distributive lattice. Then $C_k(L) \subsetneqq OF_k(L)$ if and only if L contains a proper interval which is a Boolean lattice.*

In [2] we have given a simple modular lattice M with the property that none of its proper intervals is complemented. In a simple lattice every function f: $L \to L$ is of course compatible, hence Theorem 1 cannot be generalized for an arbitrary modular lattice. In [5] R. WILLE and the author has proved the following statement:

Let L be a modular lattice of finite primitive length. Then $C_k(L) \subsetneqq OF_k(L)$ if and only if L contains a proper interval which is a Boolean lattice.

In connection to this theorem we prove the following

THEOREM 2. *There exists a modular lattice L satisfying the following conditions:*

(i) *L is the subdirect product of finite lattices;*

(ii) *there is a 1-place compatible function $\varphi \in C_1(L)$ on L which is not order-preserving;*

(iii) *none of the proper intervals $[a, b]$ of L is complemented.*

We give the proof in two steps. Let $[a, b]$ and $[c, d]$ be two isomorphic intervals of a chain C. The isomorphism f: $[a, b] \to [c, d]$ is called an interval-isomorphism, f is of course a partial operation on C, and can be extended to a unary operation $\bar{f}$ as follows: $\bar{f}(x) = f(x)$ for all $a \leqq x \leqq b$, $\bar{f}(x) = f(a)$ if $x \leqq a$ and finally $\bar{f}(x) = f(b)$ for $x \geqq b$. We say that $\bar{f}$ is the operation induced by f. The congruence relations of the partial algebra $\langle C; f \rangle$ are exactly the congruence relations of $\langle C, \bar{f} \rangle$. f is determined by $\bar{f}$, hence we can use the same letter for both.

First we construct an algebra $\mathscr{C} = \langle R, \vee, \wedge, f_i \rangle$ $i \in I$ where R denotes the bounded chain of rationals, the f_i-s are special interval-isomorphism. This will be a subdirect product of finite algebras and $\mathscr{C}$ satisfies (ii), (iii). The second step is the construction of a modular lattice L which contains $\mathscr{C}$ as a sublattice and satisfies the given properties.

2. The construction of $\mathscr{C}$

It is well-known that a bounded countable chain which is dense-in-itself is determined up to isomorphism. Therefore we can start with the chain R of rational numbers $\frac{k}{2^n}$ where $-2^n \leqq k \leqq 2^n$, $n=0, 1, \dots$. We take every $r\in R$, $r\neq \pm 1$ in two copies r and r', i.e., we split the elements. We set $1=1'$, $-1=(-1)'$.

Then we define on the set $K=\{r, r'; r\in R\}$ an ordering

(1) r is covered by r' $(r\neq \pm 1)$;

(2) $r'\leqq s$ if and only if $r\leqq s$ in R.

K is a chain and R is a subchain of K. The prime intervals of K are the following: $[r, r']$ $(r\neq \pm 1)$. The function defined by $f_{10}(r)=r+1$, $f_{10}(r')=(r+1)'$, $f_{10}(-1)=0'$ maps $[-1, 0]$ onto $[0', 1]$. f_{10} is an interval-isomorphism. On the same way we get functions f_{21}, f_{20} defined on $\left[-1, -\frac{1}{2}\right]$ resp. $\left[\left(-\frac{1}{2}\right)', 0\right]$, $f_{21}(r)=r+\frac{3}{2}$, $f_{21}(r')=\left(r+\frac{3}{2}\right)'$, $f_{21}(-1)=\left(\frac{1}{2}\right)'$; $f_{20}(r)=r+\frac{1}{2}$, $f_{20}(r')=\left(r+\frac{1}{2}\right)'$. Figure 1 helps to visualize these functions.

On the same way we can define for $n\geqq 1$, $0\leqq k<2^{n-1}$ the function f_{nk}:

$$f_{nk}\colon \left[\left(-\frac{k+1}{2^{n-1}}\right)', -\frac{k}{2^{n-1}}\right] \to \left[\left(\frac{k}{2^{n-1}}\right)', \frac{k+1}{2^{n-1}}\right],$$

$$f_{nk}(r) = r+\frac{2k+1}{2^{n-1}}.$$

If $r\in R$, $r\notin \{0, +1, -1\}$ then we define

$$g_r\colon [-r, (-r)'] \to [r, r'].$$

Finally, let φ be defined by $\varphi(1)=-1$, $\varphi(-1)=1$, $\varphi(-r)=r'$, $\varphi((-r)')=r$. The next step is the description of the congruence relations of $\langle K, \vee, \wedge, f_{nk}, g_r\rangle=\mathscr{K}_0$. First we define for each natural number $n\geqq 1$ an equivalence relation Θ_n on K: $x\equiv y(\Theta_n)$ if and only if there exists a k, $0\leqq k<2^{n-1}$ such that either $\left(\frac{k}{2^{n-1}}\right)'\leqq x$, $y\leqq \frac{k+1}{2^{n-1}}$ or $\left(-\frac{(k+1)}{2^{n-1}}\right)'\leqq x$, $y\leqq -\frac{k}{2^{n-1}}$. In Figure 1 the wavy lines denote the Θ_2-classes. There are two Θ_1-classes: $\{x, x\geqq 0'\}$ $\{x; x\leqq 0\}$. It is easy to show that Θ_n is a congruence relation of $\mathscr{K}_0$, $\mathscr{K}_0/\Theta_n$ is finite and $\bigwedge_{n=0}^{\infty} \Theta_n=\omega$. By an easy computation — applying the operations f_{nk} — we get that the principal congruence

$$\Theta\left(\left(\frac{k}{2^{n-1}}\right)', \frac{k+1}{2^{n-1}}\right)$$

is Θ_n.

Principal congruences of $\mathscr{K}_0$ are the congruence relations Θ_n and the congruence relations $\Theta(r, r')$. All these are compatible with φ, consequently φ is a congruence-preserving mapping, and φ of course is not order-preserving. $\mathcal{K}_0$ satisfies (i). $\mathcal{K}_0$ contains complemented intervals, these are the prime intervals $[r, r']$.

For each natural number i we take an isomorphic copy $\mathcal{K}_i$ of $\mathcal{K}_0$ such that $i \neq j$ implies $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset$. We put an isomorphic copy of $\mathcal{K}_1$ into the prime interval $[r, r']$ of $\mathcal{K}_0$, i.e., we have an isomorphism φ_r^1: $\mathcal{K}_1 \to [r, r']$, satisfying $\varphi_r^1(1_1) = r'$, $\varphi_r^1(0_1) = r$, where 0_1 resp. 1_1 are the zero resp. unit elements of $\mathcal{K}_1$ (see Figure 2).

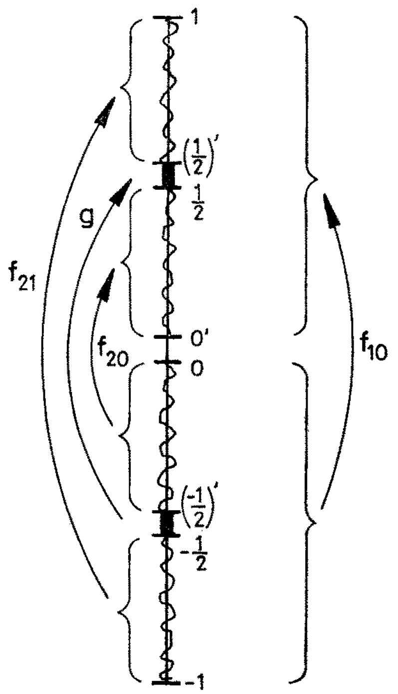

Fig. 1

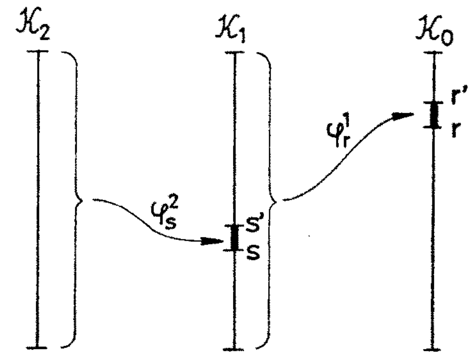

Fig. 2

Using this construction for all prime intervals of $\mathcal{K}_0$ we get a new chain C_0. φ_r^1 is an isomorphism, hence to the interval-isomorphisms of $\mathcal{K}_1$ there correspond interval-isomorphisms of $\varphi_r^1(\mathcal{K}_1) \subset C_0$. By this construction, the prime intervals of $\mathcal{K}_0$ in C_0 are isomorphic to $\mathcal{K}_1$ i.e., to $\mathcal{K}$. Continuing this construction we define the isomorphisms

$$\varphi_r^2 \colon \mathcal{K}_2 \to \mathcal{K}_1$$

then $\bigcup\limits_{r,s \in R} (\varphi_s^2(\mathcal{K}_2) \cup \varphi_r^1(\mathcal{K}_1) \cup \mathcal{K}_0)$ is a chain C_1. On this way we get a sequence of chains $C_0 \subset C_1 \subset \dots$. Let $\mathcal{C}$ be the chain $\bigcup\limits_{i=0} C_i$, i.e., the direct limit of the C_i-s.

The conditions (ii) and (iii) are obviously satisfied. We prove that $\mathcal{C}$ is the subdirect product of finite algebras. We define special congruences Φ_n on $\mathcal{C}$ $(n=0, 1, \dots)$ such that $\mathcal{C}/\Phi_n$ is finite and $\bigwedge\limits_{n=0}^{\infty} \Phi_n = \omega$.

By the isomorphism $\mathcal{K}_0 \cong \mathcal{K}_i$ the image of the congruence relation Θ_n is denoted by Θ_n^i. Let us take Θ_1^j on $\mathcal{K}_j$ for $j>n$ and Θ_n^i on $\mathcal{K}_i$ for $i \leqq n$. By the construction of $\mathcal{C}$ the image of these congruences defines a congruence relation Φ_n on $\mathcal{C}$, Φ_n is the transitive hull of all Θ_1^j and Θ_n^i $(i \leqq n < j)$. Then it is easy to show that $\wedge \Phi_n = \omega$. On the other hand from $\mathcal{K}_0/\Theta_1 \cong 2$ it follows that $\mathcal{C}/\Phi_n$ is finite.

3. The construction of the modular lattice L

We denote the chain of all non-negative rational numbers by Q^+ and Q^- is the chain of all non-positive rationals. We define a sublattice D of $Q^+ \times Q^-$. Let $A = \{(x, y);\ 0 \leqq x < 1,\ -1 < y \leqq 0\} \subseteq Q^+ \times Q^-$, $B = \{(r, -1);\ r \geqq 1\}$ and

$$C = \{(1, r);\ r \leqq -1\}.$$

Then $(Q^+ \times Q^{-1}) \setminus \{A \cup B \cup C\}$ is a sublattice D (see Figure 3).

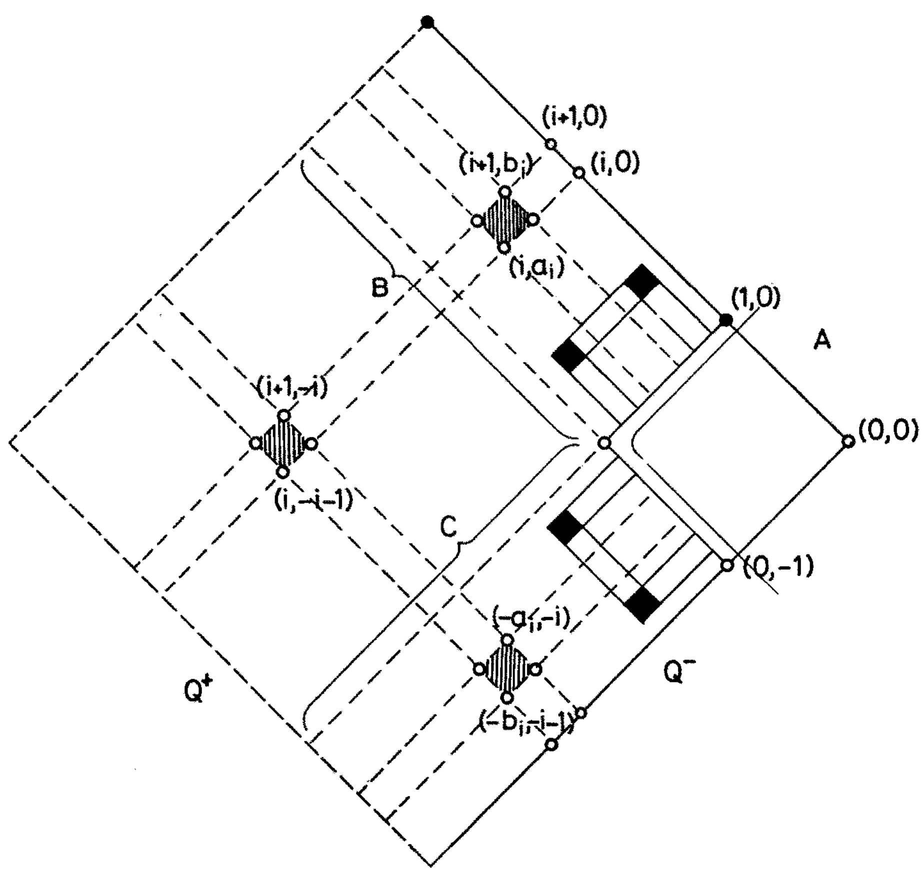

Fig. 3

The elements $\{(1, r);\ -1<r\leqq 0\}\cup\{(r, -1);\ 0\leqq r<1\}$ form a bounded countable chain which is dense-in-itself, hence we can identify this with the chain $\mathscr{C}$. If φ is the congruence-preserving function which is not order-preserving then we can assume: $\varphi((1, r))=(-r, -1)$. Then we can extend this φ to D: $\varphi(x, y)=(-y, -x)$.

Let M_3 be the five-element non-distributive modular lattice and let R be the bounded chain of rationals. Then there exists a modular lattice $M_3[R]$ having the following properties:

(a) $M_3[R]$ contains a $\{0, 1\}$-sublattice $\{0, a_1, a_2, a_3, 1\}$ isomorphic to M_3;

(b) the interval $[0, a_1]$ is isomorphic to R.

This lattice is determined up to isomorphism (see [4]).

An important property of $M_3[R]$ is that Con $(M_3[R])$ is isomorphic to Con (R). The intervals $[0, a_1]$ and $[0, a_3]$ are projective.

On $\mathscr{C}$ we have two different types of operations. Let $r=[0, 0']$. Take the operation of K_0, $\varphi_r^1(K_1)$, $\varphi_r^2\varphi_r^1(K_2)$, ... These are countable many unary operations therefore these may be enumerated, as $f_1, f_2, \dots$. Let us assume that the corresponding interval isomorphism is

$$f_i\colon\ [a_i, b_i]\to[-b_i, -a_i].$$

To i we can associate three intervals of D

$$I_{i1}=\big[(-b_i, -2(i+1)),\ (-a_i, 2i)\big]$$

$$I_{i2}=\big[(2i, -2(i+1)),\ (2(i+1), -2i)\big]$$

$$I_{i3}=\big[(2i, ai),\ (2(i+1), b_i)\big].$$

All these intervals are isomorphic to $R\times R$.

The other type of the operations are the operations of the chains K_j, $j>0$. These may be enumerated as $g_1, g_2, \dots$. Let us assume that the corresponding interval isomorphism is

$$g_i\colon\ [u_i, v_i]\to[w_i, z_i].$$

Then we can assume that $u_i, v_i, w_i, z_i\geqq 0'\in K_0\subseteqq\mathscr{C}$. Let g_i be defined by

$$g_i'\colon\ [-v_i, -u_i]\to[-z_i, -w_i].$$

To each g_i (resp. g_i') we associate the following two intervals

$$J_{i1}=[(z_i+1, v_i),\ (z_i+2, u_i)]$$

$$J_{i2}=[(2i+1, z_i),\ (2i+1, w_i)].$$

Now, we change each I_{ik}, J_{ik} to the lattice $L_{ik}\cong M_3[R]$. The elements $0\leqq x\leqq a_1$, $0\leqq y\leqq a_3$ generates a sublattice of $M_3[R]$ isomorphic to I_{ik}, i.e., I_{ik} is a sublattice of L_{ik}. This technique was developed in [3]. On this way we get from D a lattice $L(\subseteqq D)$ in which the intervals $[a_i, b_i]$, $[-b_i, -a_i]\subseteqq C$ (resp. $[u_i, v_i]$, $[w_i, z_i]$) are projective, we say that this projectivity realize the functions f_i, g_i. Then Con $(L)\cong$ $\cong$Con $(\mathscr{C})$ hence L satisfies the three conditions (i)—(iii).

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