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REMARK ON GENERALIZED FUNCTION LATTICES

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1. Introduction

G. Birkhoff has introduced the exponentation of partially ordered sets; if X, Y are partially ordered sets then Y^X denote the set of all order-preserving maps of X to Y partially ordered by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in X$. Let L be a lattice and P a partially ordered set; then L^P is a lattice, the so called function lattice. Studying the structure and decomposition of function lattices, D. DUFFUS and I. RIVAL [2] have proved the following theorem:

Let L be a finite lattice and P a finite partially ordered set with $|P|=n$. Then

$$\mathrm{Con}\,(L^P) \cong \big(\mathrm{Con}\,(L)\big)^n.$$

This theorem asserts that the congruence lattice of L^P is a direct power of Con (L). For infinite P this theorem does not remain valid, e.g. if $L \cong \underset{\sim}{2}$ (where $\underset{\sim}{2}$ denotes the two element chain) and P is the chain of rationals, then $\mathrm{Con}\,(\underset{\sim}{2}^P)$ is not a direct power of $\underset{\sim}{2}$.

The purpose of this paper is to give a generalization of this theorem for arbitrary partially ordered sets P. For this generalization we need the notion of the extension of a (finite) lattice by a bounded distributive lattice (see [5]) which generalizes the notion of the function lattice.

2. Totally order disconnected spaces

A subset E of a partially ordered set X is increasing if $x \in E$, $y \geqq x$ imply $y \in E$. Analogously we get the notion of a decreasing set. Let $(X, \mathcal{T}, \leqq)$ be an ordered space, i.e. a set X with a topology $\mathcal{T}$ endowed with the relation $\leqq$. Each set $\mathcal{U}$ consisting of the increasing sets in $\mathcal{T}$ and the set $\mathcal{L}$ consisting of the decreasing sets in $\mathcal{T}$ defines a topology on X. The triple $(X, \mathcal{T}, \leqq)$ is called totally order disconnected if given $x, y \in X$, $x \not\geqq y$, there exist disjoint $\mathcal{T}$-clopen sets $U \in \mathcal{U}$, $L \in \mathcal{L}$ such that $y \in U$, $x \in L$. (See CANFELL [1] or PRIESTLEY [3].)

Let D be a bounded distributive lattice and X the poset of all ultrafilters of D, i.e. $\leqq$ is the set-theoretical inclusion. $\mathcal{T}$ is the product topology induced from Hom $(D, \underset{\sim}{2})$ which is the set of all homomorphisms of D onto $\underset{\sim}{2}$ (i.e. $\mathcal{T}$ is the weak topology induced by Hom $(D, \underset{\sim}{2})$). Then $(X, \mathcal{T}, \leqq)$ is totally order disconnected. The main theorem of [3] assert that D is isomorphic to the dual lattice of $(X, \mathcal{T}, \leqq)$, i.e. to the lattice of all clopen increasing subsets.

Let L be an arbitrary lattice. $L[D]$ is the lattice of all continuous monotone maps of the totally order disconnected space X into the discrete space L. The constant mappings form a sublattice of $L[D]$ isomorphic to L. We identify L with this sublattice. If $a \in L$ then denote the corresponding diagonal element by $\bar{a}$. The reformulation of Priestley's theorem is the following; for every bounded distributive lattice D, $\underset{\sim}{2}[D] \cong D$ holds.

If D is finite, then it is easy to show that $L[D]$ is isomorphic to L^X, i.e. $L^X \cong L[\underset{\sim}{2}^X]$.

Let L be finite. If a/b is a prime quotient of L then the corresponding quotient $\bar{a}/\bar{b}$ of $L[D]$ is isomorphic to D.

We call $L[D]$ a generalized function lattice.

3. The congruence lattice of $L[D]$

The following theorem generalizes the result of Duffus and Rival.

THEOREM. *Let L be a finite lattice and D a bounded distributive lattice. Then*

$$\mathrm{Con}\,(L[D]) \cong (\mathrm{Con}\,(L))[\mathrm{Con}\,(D)].$$

Using this result, we can prove the theorem of Duffus and Rival as follows. Let L be a finite lattice and D a finite distributive lattice. P denotes the dual of the poset of all join-irreducible elements of D. Then $L[D]$ is the function lattice L^P. On the other hand, $\mathrm{Con}\,(D)$ is a finite Boolean algebra isomorphic to $\underset{\sim}{2}^n$, where $n=|P|$. Then $(\mathrm{Con}\,(L))[\mathrm{Con}\,(D)] \cong (\mathrm{Con}\,(L))^n$, hence $\mathrm{Con}\,(L^P) \cong (\mathrm{Con}\,(L))^n$.

PROOF. $L[D]$ is a subdirect power of L having the following two properties:

(i) $L[D]$ contains the constant mappings, i.e. the diagonal elements.

(ii) if a covers b in L then the quotient $\bar{a}/\bar{b}$ of $L[D]$ is isomorphic to D; we have a natural isomorphisms $\varepsilon_{ab}\colon \bar{a}/\bar{b} \to D$ which is the extension of the mappings $a \to 1$, $b \to 0$ $(0, 1 \in \underset{\sim}{2})$.

We will prove slightly more: if S is an arbitrary subdirect power of L satisfying (i) and (ii) then $\mathrm{Con}\,(S) \cong (\mathrm{Con}\,L)\,[\mathrm{Con}\,(D)]$.

Let θ be a congruence relation of S. Then θ_{ab} denotes the restriction of θ to the quotient $\bar{a}/\bar{b}$, where $a \succ b$ in L. $\bar{\theta}_{ab}$ denotes the extension of θ_{ab} to S, then $\bar{\theta}_{ab}$ is the smallest congruence relation of S which, restricted to $\bar{a}/\bar{b}$, is θ_{ab}.

If a/b runs over all prime quotients we get the family $\{\theta_{ab}\}$. We shall show that θ is uniquely determined by this family (i.e. $\theta \neq \Phi$ implies the existence of $a, b \in L$, $a \succ b$ such that $\theta_{ab} \neq \Phi_{ab}$). Let $u \equiv v(\theta)$, $u > v$, $u, v \in S$, i.e. $u=(u(i))$, $v=(v(i))$ where $u(i)$ resp. $v(i)$ are the i-th components ($i \in X$ and X is the set of all ultrafilters of D). Then $u(i) \geqq v(i)$ for all i. If $u(i) > v(i)$ for some i we choose the elements a, $b \in L$ such that $u(i) \geqq a \succ b \geqq v(i)$ (L is finite). Then $u \equiv v(\theta)$ implies $(u \wedge \bar{a}) \vee \bar{b} \equiv (v \wedge \bar{a}) \vee \bar{b}(\theta)$, i.e. $(u \wedge \bar{a}) \vee b \equiv (v \wedge \bar{a}) \vee b(\theta_{ab})$. The i-th components of these elements are a and b, hence the join of all $\bar{\theta}_{ab}$ is the congruence relation θ. We have therefore that θ is determined by the family $\{\Theta_{ab}\}$ where each θ_{ab} is a congruence relation on the suitable $\bar{a}/\bar{b} \cong D$.

Conversely, let $\{\theta^*_{ab}\}$ be a family of congruence relations ($\theta^*_{ab} \in \mathrm{Con}\,(\bar{a}/\bar{b})$, $a \succ b$) such that $\theta(a, b) \leqq \theta(c, d)$ $(a \succ b,\ c \succ d)$ implies $\varepsilon_{ab}\theta^*_{ab} \geqq \varepsilon_{cd}\theta^*_{cd}$ in $\mathrm{Con}\,(D)$. Then it is easy to see that there exists an "extension" $\theta \in \mathrm{Con}\,(S)$ such that the restriction of θ to $\bar{a}/\bar{b}$ is θ^*_{ab}.

$\mathrm{Con}(L)$ and $\mathrm{Con}(D)$ are distributive lattices, hence by a theorem of R. QUACKENBUSH [4] $(\mathrm{Con}(L))[\mathrm{Con}(D)]$ is isomorphic to the free product $\mathrm{Con}(L) * \mathrm{Con}(D)$ in the variety of distributive lattices. The free product is commutative, therefore we get

$$(\mathrm{Con}(L))[\mathrm{Con}(D)] \cong (\mathrm{Con}(D))[\mathrm{Con}(L)].$$

But L is a finite lattice, i.e. $\mathrm{Con}(L)$ is a finite distributive lattice. Thus if Y denotes the dual of the partially ordered set of all join irreducible elements of $\mathrm{Con}(L)$ then $(\mathrm{Con}(D))[\mathrm{Con}(L)]$ is nothing else than the function lattice $\mathrm{Con}(D)^Y$.

A join-irreducible congruence relation of L has the form $\theta(a, b)$, where a covers b. This implies that we have a one-to-one correspondence between $\mathrm{Con}(S)$ and $(\mathrm{Con}(D))^Y$ which proves our theorem.

Let L be a finite simple lattice, i.e. $\mathrm{Con}(L) \cong \underset{\sim}{2}$. Then $(\mathrm{Con})(L)[\mathrm{Con}(D)] \cong \underset{\sim}{2}[\mathrm{Con}(D)] \cong \mathrm{Con}(D)$, thus we have

COROLLARY 1. *If L is a finite simple lattice then* $\mathrm{Con}(L[D])$ *is isomorphic to* $\mathrm{Con}(D)$.

If L is a finite modular lattice then $\mathrm{Con}(L) \cong \underset{\sim}{2}^n$. Hence we get

COROLLARY 2. *If L is a finite modular lattice then* $\mathrm{Con}(L[D]) \cong (\mathrm{Con}(D))^n$ *where n is the number of irreducible congruences of L.*

PROBLEM. Does the theorem remain valid for an infinite L?

References

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