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On splitting modular lattices

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1. Introduction

A finite subdirectly irreducible algebra is splitting in a variety if there is a largest subvariety of this variety not containing it. The splitting lattices are those subdirectly irreducible lattices which are the bounded homomorphic images of finitely generated free lattices (R. McKenzie [2]). This result does not supply necessary and sufficient conditions for a splitting lattice in subvarieties of the variety of all lattices. Let $\mathscr{M}$ be the variety of all modular lattices. The description of splitting lattices in $\mathscr{M}$, i.e. of splitting modular lattices is an open problem. In this paper we give a necessary condition for a lattice S to be splitting modular.

2. Preliminaries, result

We denote the five element modular non-distributive lattice by M_3; M_3 with an additional atom is called M_4, etc. We call an ordered five-tuple (v, x, y, z, u) of elements from a modular lattice a *diamond* if these elements form a copy of M_3 with v and u as the bottom and the top elements, respectively. Two quotients a/b and c/d of a lattice L are transposes if either $a=b\vee c$ and $d=b\wedge c$ or $c=a\vee d$ and $b=a\wedge d$. The quotient a/b is said to be *projective* to c/d – in symbol $a/b\approx c/d$ – if there exists a sequence of quotients $a/b=a_0/b_0, a_1/b_1, \dots, a_n/b_n=c/d$ such that a_k/b_k and a_{k+1}/b_{k+1} are transposes for every $0\leqq k<n$. A sublattice K of L is called an *isometric* sublattice if a prime quotient in K is a prime quotient in L. An element $a\in L$ is double-irreducible if it is join- and meet-irreducible. If a is double-irreducible then $L_a=L\setminus\{a\}$ is a sublattice of L.

Theorem. *Let* (v, x, y, z, u) *be an isometric diamond of a splitting modular lattice* S. *If* y *is double-irreducible then the quotients* x/v *and* z/v *are not projective in the sublattice* $S_y=S\setminus\{y\}$.

This theorem implies

Corollary 1 (A. Day, C. Herrmann and R. Wille [1]). *M_4 is not splitting modular.*

Corollary 2. *The lattice represented by Fig. 1 is not splitting modular.*

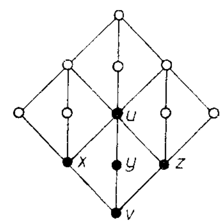

Fig. 1.

3. Function lattices

Let L be a lattice and let P be a partially ordered set. L^P denotes the lattice of all order-preserving maps of P to L partially ordered by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in P$. L^P is called function lattice and this concept is a powerful tool by the construction given in this paper. If $a \in L$ then $\bar{a}$ denotes the corresponding constant mapping, i.e. $\bar{a}(x) = a$ for each $x \in P$. If a/b is a prime quotient of L then the corresponding quotient $\bar{a}/\bar{b}$ of L^P is isomorphic to $\underset{\sim}{2}^P$, where $\underset{\sim}{2}$ denotes the two element lattice. $\underset{\sim}{2}^P$ is a distributive lattice. Obviously L^P is a subdirect power of L. The constant mappings form a sublattice of L^P which is isomorphic to L; we can identify L with this sublattice.

Consider the chain N of non-negative integers, the corresponding ordinal is denoted by ω. Similarly, ω^* is the ordinal corresponding to the chain of non-positive integers. Then using the well-known ordinal sum we get the ordinals $\omega+1$, $1+\omega^*$, $\omega+2$ where $\omega+2$ corresponds to

$$0 < 1 < 2 < \ldots < d < \infty,$$

$\omega+1$ corresponds to $0 < 1 < \ldots < \infty$, and $1+\omega^*$ corresponds to $0 > -1 > -2 > \ldots > \ldots > -\infty$. Trivially $\omega+1 \cong \underset{\sim}{2}^{\omega^*}$ and $\omega+2 \cong \underset{\sim}{2}^{1+\omega^*}$.

Let D be a filter of $1+\omega^*$ and let L be a finite lattice. If $f \in L^D$ then there exists a $-k \in D$ such that $f(-k) \leqq f(-n)$ for every $-n \in D$. We define $\bar{f} \in L^{1+\omega^*}$ as follows: $\bar{f}(-n) = f(-n)$ if $-n \in D$ and $\bar{f}(t) = f(-k)$ if $t \notin D$. Then $f \to \bar{f}$ is obviously the canonical embedding of L^D into $L^{1+\omega^*}$. If D is the filter ω^* then we get an embedding $L^{\omega^*} \to L^{1+\omega^*}$. The chain $\underset{\sim}{k} = \{0, -1, \ldots, -k\}$ is a filter of $1+\omega^*$ hence we get again an embedding $L^{\underset{\sim}{k}} \to L^{1+\omega^*}$.

Lemma 1. *Let L be a finite lattice. The ideal lattice $I(L^{\omega^*})$ is isomorphic to $L^{1+\omega^*}$.*

Proof. We have the canonical embedding $f \to \bar{f}$ of L^{ω^*} into $L^{1+\omega^*}$. Let $g \in L^{1+\omega^*}$ and take all $f \in L^{\omega^*}$ for which $\bar{f} \leqq g$. All these f-s form an ideal I_g of L^{ω^*}. It is easy to show that the correspondence $g \to I_g$ is an isomorphism between $L^{1+\omega^*}$ and $I(L^{\omega^*})$.

4. Gluing of lattices

Let A and B be two lattices with isomorphic sublattices $C \cong C'$ where $C \subseteqq A$ and $C' \subseteqq B$. We assume that A and B are disjoint. The set-theoretical union $L = A \cup B$ with C and C' identified can be made into a poset by defining $x \leqq y$ if and only if one of the following conditions is satisfied:

(i) $x \leqq y$ in A or in B;

(ii) $x \leqq c$ in A and $c' \leqq y$ in B for some $c \in A$ where c and c' are corresponding elements under the isomorphism $C \cong C'$;

(iii) $x \leqq c'$ in B and $c \leqq y$ in A where c, c' are corresponding elements.

In general, L need not be a lattice. If L is a lattice then L *is the lattice obtained by gluing together* A *and* B *by* $C \cong C'$. In the following we give a special condition for the sublattices C and C' such that L is a lattice.

A subchain C of a lattice L is called an m-subchain if the following conditions are satisfied:

(1) If $t \leqq c$ where $t \in L$, $c \in C$ then there exists a least $\bar{t} \in C$ such that $t \leqq \bar{t} \leqq c$. Similarly, if $c \leqq t$ $(c \in C)$ then we have a greatest $\underline{t} \in C$ such that $c \leqq \underline{t} \leqq t$;

(2) $$a \leqq b, \quad \bar{a} = \bar{b}, \quad \underline{a} = \underline{b} \quad \text{imply} \quad a = b;$$

(3) Let $c_1 > c_2$, $c_1, c_2 \in C$. If $c_2 \| t$ then $c_1 \vee t > c_2 \vee t$ and dually $c_1 \| t$ implies $c_1 \wedge t > c_2 \wedge t$;

(4) $r > s$, $c_1 > c_2 (c_1, c_2 \in C)$, $r \| c_i$, $s \| c_i$ imply that either $r \vee c_1 > s \vee c_1$ or $r \wedge c_2 > s \wedge c_2$;

(5) $\bar{r} = \bar{s}$, $r > s$, $c \| r, s$ $(c \in C)$ imply $c \wedge r > c \wedge s$ and dually.

A $\{0, 1\}$-subchain of a bounded lattice L is a subchain containing the 0 and 1 of L.

Lemma 2. *Let* A *and* B *be two modular lattices with isomorphic subchains* $C \cong C'$. *Let* C *be an* m-*subchain of* A *and let* C' *be a* $\{0, 1\}$ m-*subchain of* B. *Then the poset* $L = A \cup B$ *is a modular lattice.*

Proof. First we show that L is a lattice. Take two elements $a \in A$, $b \in B$, $a, b \notin C$. Then we have a $\bar{b} \in C$, $\bar{b} \geqq b$. If $a \geqq \bar{b}$ then $a \vee b = a$. If $a < \bar{b}$ in A then by (1) we have $\bar{a} \in C$ such that $a < \bar{a} \leqq b$. Take the join $\bar{a} \vee b$ in B then this element is obviously the least upper bound of a and b in L. If $a \nless \bar{b}$ then $b < \bar{a}$ hence the join $a \vee \bar{b}$ of a and $\bar{b}$ in A is the least upper bound of a and b in L.

Similarly we can prove the existence of the greatest lower bound of a and b, i.e. L is a lattice.

Let us assume that L is not modular, i.e. that L contians a pentagon with the elements $o<s<r<i$, $o<t\leqq i$. We distinguish several cases. A and B are modular lattices, hence $r, s, t\in A$ and similarly $r, s, t\in B$ is impossible.

(a) $r\in A, r\notin B$, $s, t\in B$. Then $i=s\vee t$, $s, t\in B$ imply $i\in B$. From (1) we get the existence of the elements $\bar{r}, \underline{r}\in C$ for which $s\leqq\underline{r}<\bar{r}<i$. By the modularity of B we get $s\vee(\bar{r}\wedge t)=\bar{r}\wedge(s\vee t)=\bar{r}$, i.e. $\underline{r}\vee(\bar{r}\wedge t)=\bar{r}$, and $\underline{r}\|\bar{r}\wedge t$, a contradiction to (3).

(b) $r, s\in A$, $t\in B$, $r, s, t\notin C$. Then we have the following possibilities:

(b$_1$) $o, i\notin B$. Using (1) we get $c_1=\bar{t}$ and $c_2=\underline{t}$ for which $i>c_1>t>c_2>o$. From (4) we conclude that either $r\vee c_1>s\vee c_1$ or $r\wedge c_2>s\wedge c_2$, which is a contradiction to the assumption that o, r, s, t, i form a pentagon.

(b$_2$) $o\notin B$, $i\in B$. Then we have $c_1=\bar{r}$, $c_2=\bar{o}$ for which $r<c_1\leqq i$, $o<c_2<t$. If $\bar{r}>\bar{s}$ then using the modularity of B we get $\bar{s}\vee(\bar{r}\wedge t)=\bar{r}\wedge i=\bar{r}$, $\bar{s}\|t$, a contradiction to (3), i.e. $\bar{r}=\bar{s}$. Then $c_2\|r, s$, by (5) $r\wedge c_2>s\wedge c_2$, contradiction.

(b$_3$) $o, i\in B$. Let $c_1=\bar{r}$, $c_2=\underline{s}$, then $r<c_1\leqq i$ and $s>c_2\geqq o$. From (2) we get that either $\bar{r}>\bar{s}$ or $\underline{r}>\underline{s}$. Let us assume that $\bar{r}>\bar{s}$. Then by the modularity of B we get $\bar{s}\vee(\bar{r}\wedge t)=\bar{r}\wedge(\bar{s}\vee t)=\bar{r}\wedge i=\bar{r}$, hence by (3) $\bar{r}\wedge t\in C$. But C is a chain thus $\bar{r}=\bar{r}\wedge t$, i.e. $\bar{r}\leqq t$, contradiction.

5. Proof of the theorem

Let S be a finite subdirectly irreducible modular lattice with an isometric diamond (v, x, y, z, u) such that y is a double-irreducible element. Let us assume that the quotients x/v and z/v are projective in the sublattice $S_y=S\setminus\{y\}$. We have to prove that S is not splitting modular.

First we take the function lattice $A=S_y^{1+\omega^*}$. Then the quotient u/x is a chain isomorphic to $\omega+2$, say

$$x=x_0\prec x_1\prec x_2\prec\ldots\prec x_d\prec x_\infty=u.$$

Similarly u/z is the following chain:

$$z=z_0\prec z_1\prec z_2\prec\ldots\prec z_d\prec z_\infty=u.$$

Let us take the elements: $w_0=x_1\wedge z_0$, $w_1=x_2\wedge z_1$, $\ldots$, $w_k=x_{k+1}\wedge z_k$, $\ldots$, $w_d=x_d\wedge z_d$, $w_\infty=u$. These elements form an m-subchain C of A.

Let B be a subdirect product of two copies of $\omega+2$, containing all $(a, b)\in\in(\omega+2)\times(\omega+2)$ for which $a\leqq b$. Then the elements $w_0'=(0, 0)$, $w_1'=(1, 1)$, $\ldots$, $w_k'==(k, k)$, $\ldots$, $w_d'=(d, d)$, $w_\infty'=(\infty, \infty)$ form a $\{0, 1\}$ m-subchain C' of B and C is isomorphic to C'. The lattices A and B are illustrated in Fig. 2.

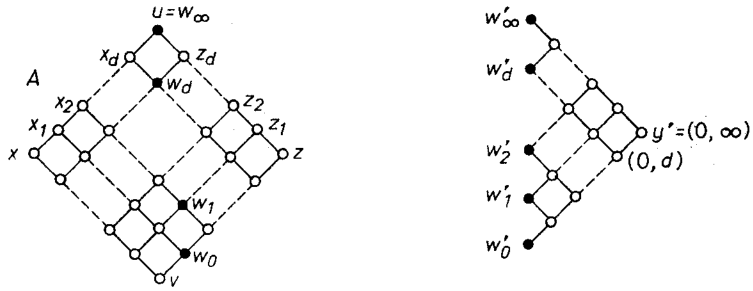

Fig. 2.

B' will denote the principal ideal $(w_d']$ of B.

Let $\overline{M}$ be the lattice obtained by gluing together A and B identifying the corresponding elements under the isomorphism $C \cong C'$. By Lemma 2 $\overline{M}$ is a modular lattice. We define some sublattices of $\overline{M}$.

If we omit all elements (k, ∞) $(k < \infty)$ from $\overline{M}$ we get the sublattice M of $\overline{M}$. In other words, M is the lattice obtained by gluing together A and B' identifying w_d and w_d', w_k' and w_k' $(k=0, 1, \ldots)$.

The next step is to define the finite sublattices M_k $(k=0, 1, 2, \ldots)$ of $\overline{M}$.

For a finite cardinal k we define A_k to be S_y^{k+1}. Then by the canonical embedding defined in section 3, A_k is a sublattice of A. The quotient u/x is a $k+2$ element chain

$$x = x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_k \prec x_\infty = u;$$

hence the elements $w_0, w_1, \ldots, w_{k-1}$ are contained in A_k. Let B_k be the principal ideal $(w_{k-1}']$ of B. Then M_k is the lattice obtained by gluing together A_k and B_k identifying the corresponding elements of the subchains $C_k = \{w_0, w_1, \ldots, w_{k-1}\}$ and $C_k' = \{w_0', w_1', \ldots, w_{k-1}'\}$. The corresponding diagram is given by Fig. 3.

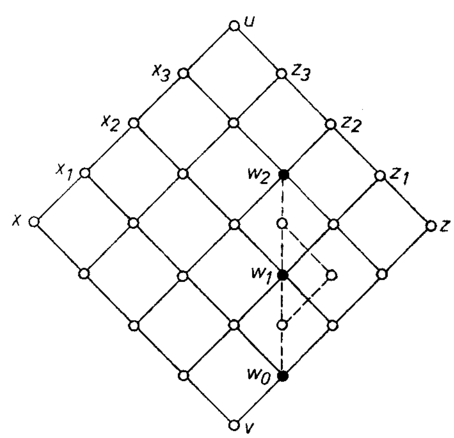

Fig. 3.

Every M_k is a sublattice of M, hence $M^* = \bigcup_{k=0}^{\infty} M_k$ is a sublattice of M. By Lemma 1, the ideal lattice of M^* is the lattice M, i.e. $I(M^*) = M$.

Lemma 3. *S is contained in the variety generated by the lattices M_k for $1 \leqq k < \infty$.*

Proof. Let $\mathscr{K}$ be the variety generated by the lattices M_k for $1 \leqq k < \infty$. Then $M^* = \bigcup_{k=0}^{\infty} M_k$ is in $\mathscr{K}$. This implies that the ideal lattice of M^* is contained in $\mathscr{K}$, i.e. $M \in \mathscr{K}$. We will prove that S is an epimorphic image of M. Therefore $S \in \mathscr{K}$.

Let θ be the congruence relation of $\omega+2$ which has exactly two congruence classes, $\{0, 1, 2, \ldots\}$ and $\{d, \infty\}$. The factor lattice is the two element lattice.

Let a/b be a prime quotient of S_y. Then there exists a natural isomorphism $\varepsilon_{ab}: \omega+2 \to a/b$, where a/b is the corresponding quotient of S_y. Then $A = S_y^{1+\omega^*}$ has a congruence relation θ_A such that the factor lattice A/θ_A is isomorphic to S_y and the restriction of θ_A to a/b is the congruence relation which corresponds to θ by the isomorphism ε_{ab}.

In the same way we get a congruence relation $\theta_{B'}$ on B' such that $\theta_{B'}$ has the classes $\{w'_d\}$, $\{x; x \in B', x \leqq w'_i$ for some $i < d\}$ and $\{(k, d); k < d\}$. Let us take the chain $\{w_0, w_1, \ldots, w_d\} \subseteqq A$. The restriction of θ_A to this chain has two classes: $\{w_0, w_1, \ldots\}$ and $\{w_d\}$. The restriction of $\theta_{B'}$ to $\{w'_0, w'_1, \ldots, w'_d\} \subseteqq B'$ has also the classes $\{w'_0, \ldots, w'_k, \ldots\}$ and $\{w'_d\}$. Let $\bar{\theta}$ be the transitive extension of θ_A and $\theta_{B'}$ to M. Then by the previous remark $\bar{\theta}|A = \theta|A$ and $\bar{\theta}|_{B'} = \theta|_{B'}$, $A/\theta_A \cong S_y$, $B'/\theta_{B'} \cong 2$. Thus we get that $M/\bar{\theta}$ is isomorphic to S, which proves our Lemma.

Let $\{M_k\}^e$ be the variety generated by M_k. The subdirectly irreducible lattices of a variety generated by a finite lattice F are epimorphic images of sublattices of F. To prove that S is not splitting we need to prove

Lemma 4. *S is not contained in the variety generated by M_k.*

Proof. Let us take the quotient u/v of S and the corresponding quotient u/v of M_k. It can be easily seen that u/v is not an epimorphic image of a sublattice of u/v, using the assumption that x/v and z/v are projective in S_y. (See [1]). This involves that M_k doesn't contain a sublattice T such that S is an epimorphic image of T.

6. Planar lattices

Let $\mathscr{K}$ be a variety of lattices. A lattice L in $\mathscr{K}$ is called finitely $\mathscr{K}$-projected if for any surjective $f: A \twoheadrightarrow L$ in $\mathscr{K}$ there is a finite sublattice of A whose image under f is L. In [3] the finitely projected planar modular lattices are characterized. From this characterization we get, using the concept of the diamond circle [4]:

Corollary 3. *A subdirectly irreducible planar modular lattice S is splitting modular if and only if S does not contain a diamond circle or a sublattice isomorphic to M_4.*

A planar modular lattice is 2-distributive. If S is 2-distributive then the lattice $\overline{M}$ is again 2-distributive. Hence we have

Corollary 4. *A subdirectly irreducible planar modular lattice S is splitting in the variety of all 2-distributive lattices if and only if S does not contain a diamond circle or a sublattice isomorphic to M_4.*

Remark. The same proof gives the following generalization of our Theorem:

Let (v, x, y, z, u) be an isometric diamond of a splitting modular lattice S and let $t \in S$ be such that $u \wedge t = v$, y is $\vee$-irreducible and $y \vee t$ is $\wedge$-irreducible. Then $S' = \{x \in S;\ x \notin {}^{y \vee t}/y\}$ is a sublattice of S and x/v, z/v are not projective in this sublattice.

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