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ON THE VARIETY GENERATED BY ALL MODULAR LATTICES OF BREADTH TWO

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R. Freese [1] has proved that the variety B_2 generated by all modular lattices of breadth less than or equal to 2 is not generated by the finite dimensional members. In this paper we show that B_2 is not generated by their member which have a property implying that no quotient is projective to its proper subquotient.

We say that the quotient c/d is weakly projective into a/b (in symbols $c/d \approx_w a/b$) in a lattice L if there is a unary algebraic function $f(x)$ over L such that $f(a) = c$ and $f(b) = d$. If $f\colon a/b \rightarrow c/d$ is one-one mapping onto c/d then a/b and c/d are called projective quotients and f is the corresponding projectivity. A finite dimensional modular lattice has obviously the following property:

(1) $c/d \approx_w a/b$ and $c \geqslant a > b \geqslant d$ imply $c = a$ and $d = b$,

i.e. no quotient is weakly projective to a proper subquotient. (It is easy to see that (1) is equivalent to the condition that no quotient is projective to a proper subquotient). In this paper we define first an other property, related to (1) which imply (1).

DEFINITION. Let $b = x_0 < x_1 < \ldots < x_n = a$ and $d = y_0 < \ldots < y_n = c$ be two chains of a lattice L and let $j_1, j_2, \ldots, j_n$ be a permutation of the integers $1, 2, \ldots, n$ such that for each $i = 1, 2, \ldots, n$, y_{j_i}/y_{j_i-1} is weakly projective into x_i/x_{i-1}. In this case we say that c/d is weakly subprojective into a/b. Let us define a lattice L to have the subprojectivity property if whenever a quotient c/d is weakly subprojective into a subquotient a/b, then $a/b = c/d$.

For $n = 1$ we get the condition (1). It is easy to give an example for a lattice which has the property (1) but fails the subprojectivity property.

THEOREM. *B_2 is not generated by the members having the subprojectivity property.*

The proof is based on [1]. To the proof we need some preliminaries.

LEMMA [2]. *Let N be a bounded distributive lattice. Then there exists a* *modular lattice K such that*

(i) *K has three elements x,y,z such that 0,x,y,z,1 form a sublattice of K isomorphic to M_5, and x/0, y/0, and z/0 are isomorphic to N.*

(ii) *Every congruence on K is determined by its restriction to x/0.*

Let K be the lattice obtained from this lemma with N equal to the bounded rationals. We take this K in five isomorphic examplars K_i, $i = 1,2,\ldots,5$ and apply the Hall-Dilworth construction ([2], [3]) to get a lattice L given with the following schematic diagram (where v_i is the least and u_i is the greatest element of K_i) (For the exact description of L we refer to [1] or [3].).

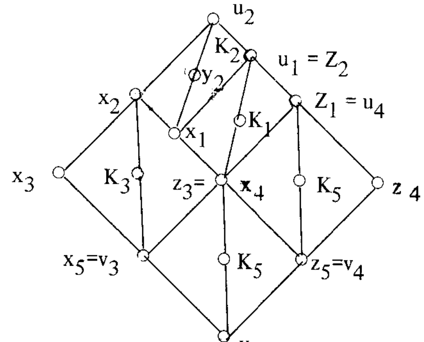

Figure 1

Let g denote a homomorphism from a modular lattice M onto L. R. Freese has proved that M contains a sublattice diagrammed in Figure 2, such that $g(\underset{\sim}{a}) = x_2$, $g(\underset{\sim}{b}) = z_1$, $g(\underset{\sim}{1}) = u_2$, $g(\underset{\sim}{c}) = v_4$, $g(\underset{\sim}{x_2}) = x_2$, $g(\underset{\sim}{y_2}) = y_2$, $g(\underset{\sim}{z_2}) = z_2$, $g(\underset{\sim}{x_1}) = x_1$ etc. and $\underset{\sim}{a}/\underset{\sim}{c}$ is projective to $\underset{\sim}{b}/\underset{\sim}{c}$.

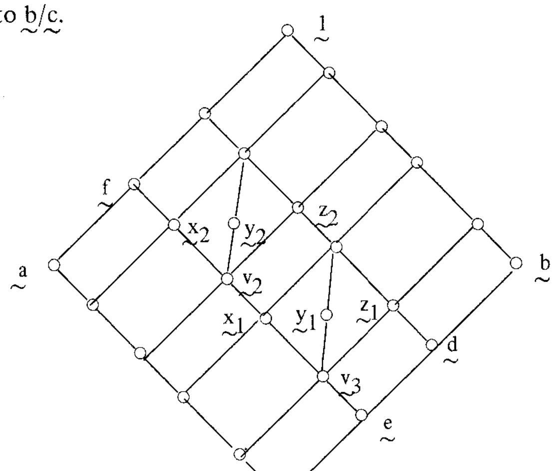

Figure 2

Some of the quotient may collapse, but the two copies of M_5 must be nondegenerate.

In the lattice L, x_2/z_3 is (weakly-) projective to the proper subquotient x_1/z_3 hence L do not have the subprojectivity property. We prove that the lattice diagrammed in Figure 2 fails to have the subprojectivity property in a special subvariety of B_2.

a/c and b/c are projective, hence f/e and b/c are projective. Let p be the corresponding projectivity: $p(b) = f$ and $p(c) = e$. Let d' and e' be the images of d and e by p, i.e. $d' = p(d)$, $e' = p(e)$.

x_1/v_1 and x_2/v_2 are projective too, the corresponding projectivity is: $g(x) = (((x \vee y_1) \wedge z_1) \vee y_2) \wedge x_2$. Then $g(x_1) = x_2$, $g(v_1) = v_2$.

We define: $e'' = g(e')$ and $d'' = (d' \vee v_2) \wedge x_2$. Both are in the quotient x_2/v_2. We shall prove that $e'' \geqslant d''$.

Let $\boldsymbol{K}$ be the variety generated by L all breadth two modular lattices having the subprojectivity property. Then $\boldsymbol{K}$ is a subvariety of B_2. We shall show that $L \notin \boldsymbol{K}$. $L \in \boldsymbol{K}$ would imply that L is the homomorphic image of a lattice T which is a sublattice of a direct product of breadth two modular lattices having the subprojectivity property. Let g denote the homomorphism $T \to L$. Then T contains the sublattice given by Figure 2. T is also a sublattice of a direct product. Let π denote the projection homomorphism onto one of these breadth two lattices. As in [1] we can assume that $\pi(a)$ and $\pi(b)$ are incomparable. $\pi(T)$ is a breadth two lattice hence the quotients $\pi(a)/\pi(c)$ and $\pi(b)/\pi(c)$ are chains, i.e. $\pi(d'')$ and $\pi(e'')$ are comparable. Hence either $\pi(e'') \geqslant \pi(d'')$ or $\pi(e'') < \pi(d'')$. We prove first that the second inequalities is impossible. We distinguish several cases.

(1) If $\pi(e') \geqslant \pi(x_1)$ then $\pi(e'') = \pi(x_2) \geqslant \pi(d'')$.

(2) If $\pi(e') \leqslant \pi(v_1)$ then $\pi(e'') = \pi(v_2)$. $\pi(d'') > \pi(e'')$ would imply that $\pi(d') > \pi(v_2)$. The quotients $\pi(d')/\pi(e')$ and $\pi(x_1)/\pi(v_1)$ are projective, and $\pi(d') > \pi(v_2) \geqslant \pi(x_1) \geqslant \pi(v_1) \geqslant \pi(e')$. This is a contradiction to the assumption that $\pi(T)$ has the subprojectivity property. We have also $\pi(e'') \geqslant \pi(d'')$.

(3) If $\pi(d') \leqslant \pi(v_2)$ then $\pi(d'') = \pi(v_2)$ i.e. $\pi(e'') \geqslant \pi(d'')$.

(4) By 1,-3, we can assume that $\pi(x_1) > \pi(e') > \pi(v_1)$ and $\pi(d') \geqslant \pi(v_2)$ (Figure 3.). If $\pi(e'') < \pi(d'')$ then we have in tne quotient $\pi(e'')/\pi(e')$ the chain $\pi(e') < \pi(x_1) \leqslant \pi(v_2) < \pi(e'')$. Applying $g^{-1}(x) = (((x \vee y_2) \wedge y_1) \vee z_1) \wedge x_1$ in $\pi(T)$ we get $g^{-1}(\pi(v_2)) = \pi(v_1)$ and $g^{-1}(\pi(e'')) = \pi(e')$. Thus we get with the unary algebraic function $\overline{p} = hp$, where $h(x) = (x \vee y_1) \wedge d$ ($\overline{p}$ is the projectivity between x_1/v_1 and d'/e' i.e., the following holds $\overline{p}(x_1) = d'$ and $\overline{p}(v_1) = e'$). $\pi(e') = \overline{p}(\pi(v_1)) \leqslant \overline{p}\pi(e')) \leqslant \overline{p}\pi(x_1)) = \pi(d')$. But by 3, $\pi(d'') \leqslant \pi(d')$ hence $\pi(d')/\pi(e')$ is weakly subprojective into $\pi(e'')/\pi(e')$, i.e. $\pi(T)$ don't have the subprojectivity property. We have also that in all cases $\pi(e'') \geqslant (d'')$. In [1] it is proved that $\pi(e'') \geqslant \pi(d'')$ imply that $g(x_2) = x_2 = v_2 = g(v_2)$ in T, which is obviously a contradiction, to the assumption that $L \in$ ***K***.

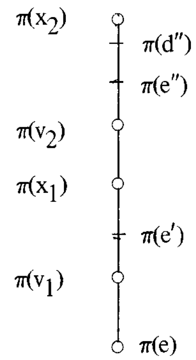

Figure 3

PROBLEM 1. Let V be a class of all breadth two modular lattices having the subprojectivity property. Is V a variety? (i.e. is V homomorphically closed?)

PROBLEM 2. Let ***K*** be the variety generated by V in B_2. Is this variety generated by its finite dimensional members?

REFERENCES

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