# On the Characterization of the Congruence Lattices of Lattices

by

E.T. SCHMIDT

It is old outstanding problem of the lattice theory to characterize the congruence lattices of lattices, it seems likely that every distributive algebraic lattice is the congruence lattice of any lattice. I shall give in this paper a brief survey of this topic and I try to sketch some ideas which can be probably help for the solution of the problem.

## §.1. Characterization Problem

We denote by $Con(K)$ the congruence lattice of the lattice $K$. We want to construct to a given distributive algebraic lattice $L$ a lattice $K$ such that $L \cong Con(K)$. By such a construction it is necessary for us to take only the compact elements of the algebraic lattice $L$. These ones forms a semilattice $L^*$ with zero and $L$ is isomorphic to the lattice of all ideals of $L^*$. The compact elements of $Con(K)$ are called compact congruence relations. The characterization problem can be formulated as follows:

For a given distributive semilattice[1) ] $F$ with zero find a lattice $K$ such that the semilattice of all compact congruence relations should be isomorphic to $F$.

First of all we take some special cases for $F$.

If $F$ is a Boolean-semilattice (i.e. a Boolean-algebra respects the join-operation) then $F$ is a lattice too and in this case $Con(F) \cong I(F)$ (lattice of all ideals), i.e. for the lattice $K$ we can choose $F$ itself.

Let $F$ be an arbitrary finite distributive (semi-) lattice. Then $F$ is determined by the poset $P$ of all join-irreducible elements of $F$ whcich are greater then $0$, $F$ is isomorphic to the lattice of all 0-ideals of $P$. It is necessary to show that for any poset $P$ there exists a lattice $K$ such that $Con(K) \cong O(P)$ (where $O(P)$ is the lattice of 0-ideāls of $P$ ). Such a lattice was constructed by G. Grätzer and E.T.Schmidt [2] (firstly by R.P. Dilworth unpublished). The construction is based on the six-element lattice given by Fig.1.

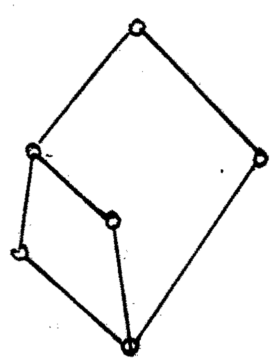

Fig.I.

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1) i.e. if $c \leq a \vee b$ then there exist $a' \leq a$, $b' \leq b$ such that $c = a' \vee b'$.

In this lattice $\Theta_{o,a} < \Theta_{o,b}$ and hence the congruence lattice is the three-element chain. The main idea of the construction given in [2] was the following: for every pair $a_i, b_i \varepsilon P$, $a_i < b_i$ we take a copie of this six-element lattice and we "hooked together" to become a lattice $K$.

This construction yields that each finite distributive lattice is isomorphic to the lattice of all congruence relations of some finite lattice.

The given construction works not only for finite distributive lattices, we can represent all distributive algebraic lattices with the property that each element is a join of completely join-irreducible elements (On other words distributive lattices which are determined by the join-irreducible elements.)

For the solution of the general problem we need an other construction.

Let $D$ be a sublattice of the lattice $K$. What is the connection between $Con^*(D)$ and $Con^*(K)$ ? In the general case the connection is obviously very loose. If $\bar{\Theta}$ denotes the congruence relation of $K$ generated by the congruence $\Theta \varepsilon Con^*(D)$ then $\Theta \to \bar{\Theta}$ is a homomorphism of the semilattice $Con^*(K)$, *i.e.* for $\Theta, \Phi \varepsilon Con^*(D)$ we have $\overline{\Theta \vee \Phi} = \bar{\Theta} \vee \bar{\Phi}$. Now, let us start with a lattice $D$ which has a "good" congruence-lattice, for instance let $D$ be a distributive lattice. The semilattice $Con^*(D)$ is then a generalized Boolean lattice (i.e. a relatively complemented distributive (semi-)lattice with zero). But the free semilattices with zero are generalized Boolean algebras, and it is easy to see that every semilattice with 0 is the homomorphic image of a semilattice $Con^*(D)$ for some $D$. Let $L$ be a homomorphic image of $Con^*(D)$. We shall construct an extension $K$ of $D$ (i.e. $D$ is a sublattice of $K$ ) such that the mapping $\Theta \to \Theta$ should be a homomorphism of $Con^*(D)$ onto $Con^*(K)$ (i.e. every congruence relation of $K$ is determined by its restriction to $D$ ) and $Con^*(K)$ should be isomorphic to $L$.

The previous observations are valid for arbitrary universal algebras. For lattices we have more, $Con^*(K)$ is a distributive semilattice and so we must have special $\Theta \to \Theta$ homomorphism. The most important property of this homomorphism is the following:

Proposition. Let $D$ be a convex sublattice of the lattice $K$. If $\Phi \vee \Psi = \Theta$, $\Theta \leq \Theta_1$ $(\Phi, \Psi, \Theta_1 \varepsilon Con^*(D))$ and $\Theta = \Theta_1$ then there exist $\Phi_1 \geq \Phi$, $\Psi_1 \geq \Psi$ $(\Phi_1, \Psi_1 \varepsilon Con^*(D))$ such that $\Phi_1 \vee \Psi_1 = \Theta_1$.

Proof. $\Theta_1$ is a compact congruence relation, i.e. $\Theta_1 = \bigvee_{i=1}^{n} \Theta_{a_i b_i}$ $(a_i < b_i)$. From $\Phi \vee \Psi = \Theta$ and $\Theta = \Theta_1$ it follows that for every $i$ $(1 \leq i \leq n)$

$$a_i \equiv b_i(\Phi \vee \Psi). \qquad (a_i, b_i \varepsilon D)$$

We have therefore the finite chains:

$$a_i = c_{o,i} < c_{1,i} < \dots < c_{n_i,i} = b_i \qquad (1 \leq i \leq n)$$

such that $c_{j,i} \equiv c_{j+1,i}$ by $\Phi$ or $\Psi$ and $c_{j,i} \varepsilon D$. Let $\Phi_1$ be the join of all congruences $\Phi \vee \Theta_{c_{j,i}, c_{j+1,i}}$ $(\varepsilon Con^*(D))$ with the property that $c_{j,i} \equiv c_{j+1,i}$ $(\Phi)$. On the same way we get $\Psi_1$. Both are obviously compact and $\Phi_1 \geq \Phi$, $\Psi_1 \geq \Psi$ $\Phi_1 \vee \Psi_1 = \Theta_1$, and $\Phi_1 = \Phi$, $\Psi_1 = \Psi$.

The proposition suggest the following definition:

<u>Definition.</u> The congruence relation $\Theta$ of the semilattice $F$ is called weak-distributive iff $u \equiv xvy\ (\Theta)$, $u \geq xvy$ imply the existence of elements $\bar{x} \geq x$ , $\bar{y} \geq y$ such that $\bar{x} \equiv x(\Theta)$; $y \equiv \bar{y}(\Theta)$ and $\bar{x}v\bar{y} = u$, (Fig.2). A homomorphism $\varphi$ is called weak-distributive if the congruence relation induced by $\varphi$ is weak-distributive.

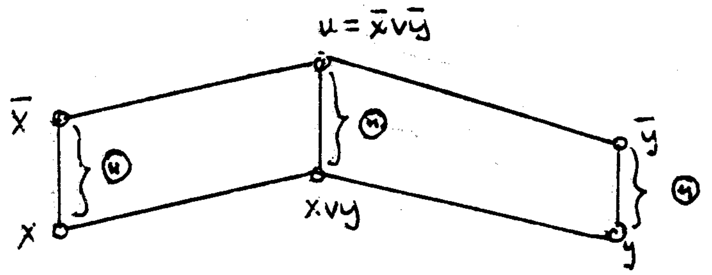


Fig.2.

<u>Examples.</u> Let $F$ be a finite distributive semilattice. Then $F$ is the homomorphic image of a Boolean semilattice $B$ : let $B$ be the Boolean-algebra generated by $F$ , then for every $x \varepsilon B$ there exists a smallest $\bar{x} \varepsilon F$ such that $x \leq \bar{x}$ . The mapping $x \rightarrow \bar{x}$ is weak-distributive homomorphism. If $F$ is a Boolean-semilattice and $p$ is irreducible element (atom) of $F$ then $\Theta\, p,a$. It is easy to prove that if $\Theta$ is a weak-distributive congruence of a distributive semilattice

$F$ then $F/\Theta$ is distributive too. An other important property is that the join of weak-distributive congruences is weak-distributive.

It is easy to give an example for a semilattice $F$ and $a,b\in F$ that there is no smallest weak-distributive congruence $\Theta$ such that $a \equiv b(\Theta)$ i.e. $\Theta_{ab}$ does not exist. The compact congruences of semilattices or distributive lattices have the property that every congruence class has a maximal element.

Definition 2. A congruence relation $\Theta$ of a semilattice is called monomial if every $\Theta$-class has a maximal element.

Every congruence relation of a lattice is the join of compact congruence relations, therefore it is natural to introduce the following notion:

Definition 3. A congruence relation $\Theta$ of a semilattice $F$ is called distributive if $\Theta$ is the join of weak-distributive monomiale congruences. A homomorphism $\varphi$ is called distributive if the congruence relation induced by $\varphi$ is distributive.

Let $D$ be a sublattice of $K$. If $H$ is a finite subset of $K$, then we can take $D_H = D \quad H \leq K$ as a relative sublattice of $K$. Let $\Theta_H$ be denote the congruence relation of $D_H$ induced by $\Theta \in Con^*(D)$. The congruence relation of $Con^*(D)$ induced by the mapping $\Theta \to \Theta_H$ is a monomiale distributive congruence. If $H$ runs over all finite subset of $K$ we get from Proposition

<u>Theorem 1.</u> Let $D$ be a convex sublattice of the lattice $K$ such that every congruence relation of $K$ is determined by its restriction to $D$. The homomorphism $\underline{\Theta} \to \bar{\Theta}$ $(\Theta \varepsilon Con^*(D))$ is a distributive homomorphism of $Con^*(D)$ onto $Con^*(K)$.

In [6] I have try to solve the characterization problem on the following way:

(A) To construct for an arbitrary distributive semilattice $F$ with zero a generalized Boolean-semilattice $B$ such that $F \cong B/\Theta$ where $\Theta$ is a distributive congruence;

(B) To prove that for every Boolean-semilattice $B$ and distributive congruence $\Theta$, the corresponding factorsemilattice $B/\Theta$ is isomorphic to the semilattice of all compact congruences of a lattice.

The problem (B) is solved, the first problem is still open.

<u>To problem (B)</u> we have the following

<u>Theorem 2.</u> Let $B$ be a generalized Boolean-semilattice and let $\Theta$ be a distributive congruence of $B$. The semilattice $B/\Theta$ is isomorphic to the semilattice of all compact congruence relations of a lattice.

By Theorem 1 and Theorem 2 we have:

For a semilattice $F$ the following two condition are equivalent:

(i) $F$ is the distributive homomorphic image of a generalized Boolean semilattice.

(ii) there exist a lattice $K$ and a convex distributive sublattice $D$ of $K$ such that every congruence relation of $K$ is determined by its restriction to $D$ and $Con^*(K) \cong F$.

Applications. Let $F$ be relatively pseudocomplemented lattice and take the Boolean-algebra generated by $F$ . Than we have for every $x \varepsilon B$ a smallest $\bar{x} \varepsilon F$ such that $x \leq \bar{x}$ and $x \rightarrow \bar{x}$ is a weak-distributive homomorphism of $B$ onto $F$ . Let $\Theta$ be the (join-) congruence induced by this mapping then every $\Theta$-class has a maximal element: the class containing $x$ has the maximal element $\bar{x}$ i.e. we have a distributive homomorphism . Therefore we can apply our Theorem 2, hence $F$ is isomorphic to the semilattice of all compact congruence relations of a lattice.

Let $F$ be a relatively pseudoelement semilattice. In this case is more complicated to apply Theorem 2. For every $a \varepsilon F$ the principal ideal $(a]$ is a pseudocomplemented lattice and the elements in the form $b*a$ $(b$ rans over the elements of $(a]$ , form a Boolean-algebra $B(a)$. Let $B$ be the direct product of all $B(a)$ then $F$ is a distributive homomorphic image of $B$ . This homomorphism has the property given in Theorem 2, hence we get:

Corollary. Every relatively pseudocomplemented semilattice with zero is isomorphic to the semilattice of all compact congruence relations of a lattice.

To Problem (A)

We have also the open question (A). My conjecture is, that every distributive semilattice with zero is the distributive homomorphic image of a generalized Boolean-semilattice. We give an example, that the distributive homomorphic image of a Boolean-semilattice need not to be relatively pseudocomplemented.

Let $A_o$ be an arbitrary distributive lattice with unit and without zero element and take the direct product with the two element lattice i.e. $A_o \times 2$. If we add to this a zero element 0 than we get a lattice $A$. $A$ is not relatively pseudocomplemented, the element $u=(1,0)$ has no pseudocomplement. Take the Boolean-algebra $B_o$ generalized by $A$. Let $a'$ be the relative complement of $(a,0)$ $(a\varepsilon A_o)$ in the intervall $[0,u]$ and we denote by $T$ the set of all $a'$. $B$ is the sub-Boolean-algebra generated by the elements $a$ and $a'$ $(a\varepsilon A_o)$. Let $\Theta$ be the join-congruence relation of $B$ generated by $T$ i.e. the smallest join congruence relation with the $\Theta$-class $T$. It is easy to prove that $B/\Theta$ is isomorphic to $A$, which is also isomorphic to the semilattice of all compact congruences of a lattice.

In this example the element $u$ has no psuedocomplement and therefore we have to take $u$ in "many" examplars, more precisely for every $a$ with $avu=1$ we have $a'=u(a)$ for which $ava'=1$. The distributive homomorphism $B \longrightarrow A$ maps these $a'$ on $u$.

It is easy to see that by the problem (A) we need only to take semilattices with zero and unit having the following property:

for every $a(\neq 0,1)$ there exists a finite sequence $x_1,\dots,x_n \varepsilon F$ such that $a||x_1,\dots,avx_1v\dots x_{i-1}||x_i$ $(i=1,\dots,n)$ and $avx_1v\dots vx_n=1$.

The $n$-tupel $(x_1,\dots,x_n)$ is called a generalized semicomplement of $a$. For every $a\varepsilon F$ and semicomplement $(x_1,\dots,x_n)$ take the symbol $a(x_1,\dots,x_n)$. My conjecture is that we can the Boolean-algebra $B$ generated by these symbols.

Special lattices A lattice $K$ is sectionally complemented (or principally complemented) if has a 0 and all intervals $[o,a]$ are complemented. If $\Theta$ is a congruence relation such a lattice and $[o]\Theta$ denote the $\Theta$-class containing 0 then $\Theta \to [o]\Theta$ is a one-to-one correspondence between congruences and certain ideals (which are namely kernel of congruences). The construction of [6] gives a sectionally complemented lattice i.e. we have: every finite distributive lattice is the congruence lattice of a sectionally complemented lattice. It can be proved the following stronger form of Theorem 2. $B/\Theta$ is isomorphic to the semilattice of all compact congruence relations of a lattice $K$ with the property: $K$ has an element $a$ such that every congruence relation of $K$ is determined by the congruence class contining $a$.

## §.2. Characterization of the Congruence Lattices of Special Lattice Varieties

The characterization of congruence lattices of distributive lattices is quite easy, they are the ideal-lattices of sectionally complemented distributive lattices. The characterization problem for modular lattices is open. My conjecture is that every distributive algebraic lattice is isomorphic to the congruence lattice of a modular lattice. The first step to the solution of this conjecture is the following:

Theorem 3 [7] . Every finite distributive lattice is isomorphic to the congruence lattice of a modular lattice.

By the proof of this theorem we need a special lattice and a special lattice construction.

The special lattice is the following: let $Q$ be the chain of all rational numbers $r$ , $0 \leq r \leq 1$ and let $M_3$ the five-element modular, non-distributive lattice (Fig.3). We put $Q$

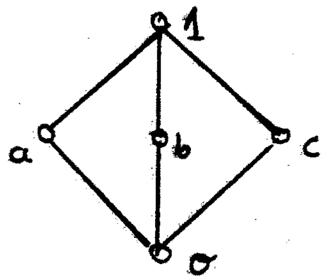


Fig.3.

in the interval $[o,a]$ i.e. we identify the rational numbers 0 and 1 with 0 and $a$. Then the set-theoretic union of $Q$ and $M_3$ is a partial lattice if we define $xvy$ only for $x,y\varepsilon M_3$ or $x,y\varepsilon Q$. Let $M$ be the modular lattice freely generated by this partial lattice. $M$ is a subdirect power of $M_3$ i.e. $M$ is in the variety generated by $M_3$. Many interesting characterizations are known of $M$. (See A.Mitschke and R. Wille [5], E.T. Schmidt [6]).

The lattice construction applied by the proof of Theorem 3 is due to R.P. Dilwórth and M. Hall [3]. Let $I_1$ be a dual-ideal of the lattice $L_1$ and let $I_2$ an ideal of the lattice $L_2$. If $I_1$ and $I_2$ are isomorphic and $x \to x'$ $(x\varepsilon I_1,\ x'\varepsilon I_2)$ is given isomorphism, then we can define a new lattice $L$: the elements of $L$ are the elements of the set-theoretical union $L_1 \cup L_2$ and we identify every $x\varepsilon I_1$ with $x'$. The ordering in $L_i \subseteq L$ has an unchanged meaning and $x \leq y,\ x\varepsilon L_1,\ y\varepsilon L_2$ iff there exists a $z\varepsilon L_1 \cap L_2$ such that $x \leq z$ in $L_1$ and $z \leq y$ in $L_2$. This construction will be referred to as the Hall-Dilworth construction. (See R. Freese, Some varieties of modular lattices not generated by their finite dimensional members, Preprint). The lattices constructed in [7] are lattices applying the Hall-Dilworth construction for isomorphic examplars of $M$, so we get a stronger version of Theorem 3:

<u>Theorem 4.</u> Let $V$ be a variety of modular lattices with the following two properties

(1) $V$ contains a non-distributive lattice;

(2) $V$ is closed under the formation of Hall-Dilworth construction.

Then every finite distributive lattice is isomorphic to the congruence lattice of $K \varepsilon V$.

<u>Corollary.</u> Every finite distributive lattice is the congruence lattice of a 2-distributive lattice.

R.S. Freese has proved (unpublished) the following sharper version of Theorem 3:

<u>Theorem 5</u>. Every finite distributive lattice is isomorphic to the congruence lattice of a finitely- generated modular lattice.

R.S. Freese pointed out that the lattices given in [7] have breath two.

The modular lattices have the following important properties: let $f(x)$ be a unary algebraic function over a modular lattice $M$: if $a < b$ $(a,b \varepsilon M)$ then there exist $a_o, b_o$ $a < a_o \leq b_o \leq b$ such that the intervals $[a_o, b_o]$ and $[f(a), f(b)]$ are isomorphic. Hence every unary algebraic function $f(x)$ define an isomorphism between $[a_o, b_o]$ and $[f(a), f(b)]$. We can take $f$ as a partial operation defined on $[a_o, b_o]$. More generally, a $*$-operation $f$ is a partial operation defined on a lattice $L$ with domain $[a,b]$ such that $x \longrightarrow f(x)$ is an isomorphism between $[a,b]$ and $[f(a), f(b)]$.

In [7] it was proved:

Let $Q$ be the chain of rational numbers $r, 0 \leq r \leq 1$.
For every finite distributive lattice there exists a partial algebra defined on $Q$

$$= < O;\ V,\ \Lambda, f_1, f_2, \ldots >$$

where $f_i$ are $*$-operations such that the congruence lattice of $Q$ is isomorphic to $L$.

In an other paper [8] I have proved:

Theorem 7. Let $K$ be an arbitrary distributive lattice. If $f_i\ (i \varepsilon I)$ are $*$-operations on $K$ then there exists a modular lattice $M$ such that the congruence lattices of $M$ and the partial algebra $<K; V, \Lambda, f_i | i \varepsilon I>$ are isomorphic.

We can formulate the following conjecture: for every distributive algebraic lattice $L$ there exists a distributive lattice $M$ and $*$-operations $f_i (i \varepsilon I)$ on $M$ such that $L$ is isomorphic to the congruence lattice of the partial algebra $<M; V, \Lambda, f_i | i \varepsilon I>$.

An other open problem is the following:

Problem. Is every distributive algebraic lattice isomorphic to the congruence lattice of a relatively complemented modular lattice?

As to finite distributive lattices the proof of the next statement can be helpfull to the solution of this problem:

Let $\mathcal{M}$ be the smallest variety of modular lattices with the following porperties (see Theorem 4):

(i) $\mathcal{M}$ contains $M_3$;

(ii) $\mathcal{M}$ is closed under formation of Hall-Dilworth construction.

Is every member of $\mathcal{M}$ a sublattice of a complemented modular lattice?

A Boolean-semilattice is called homogeneous if every two non-trivial ideals are isomoprhic. For instance the Boolean-semilattice generated by $Q$ is homogeneous. Theorem 6 can be formulated as follows: every finite distributive lattice is the distributive homomorphic image of a homogeneous Boolean-semilattice. The following question is open:

Problem. Is every distributive semilattice with zero the distributive homomorphic image of a homogeneous Boolean-semilattice?

## §3. On the Length of Lattices with Given Congruence Lattices.

J. Berman has proved [1] that for every finite chain $L$ there exists a lattice $K$ of length five such that $Con(K) \cong L$. In [9] I have generalized Berman's construction:

Theorem 8. Let $L$ be a finite distributive lattice with irreducible unit element. Then there exists a finite lattice $M$ such that $Con(M) \cong L$ and $M$ has length 5.

Problem. Does there exist to every finite distributive lattice $L$ with $n$ dual atoms a natural number $\varphi(n)$ such that $L \cong Con(K)$ for some finite lattice $K$ of length $\varphi(n)$? (Conjecture $\varphi(n)=5n$.)

## §.4. Some Related Questions

It is an open problem to show that each algebraic lattice is isomorphic to the congruence lattice of some algebra having only finite many operations. A big step toward the solution of this problem was taken in W.A. Lampe (preprint) where it was shown that any algebraic lattice whose unit element is compact is isomorphic to the congruence lattice os some grupoid. A part of this problem is the following open question:

Problem. Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebra of finite tpye?

Let $F$ is a relatively pseudocomplemented lattice. Denote $B$ the Boolean-algebra generated by $F$ . If $x \in B$ than $\bar{x}$ is the smallest element of $F$ with the property $x \leq \bar{x}$ . $f(x)=\bar{x}$ is a unary operation. Denote $x'$ the complement of $x \in B$. The algebra $\mathcal{A} =< B; V', f >$ is of typ (2,1,1) and it is easy to see that $Con(\mathcal{A}) \cong I(F)$.

An other related problem is the following:

Problem. Does there exist a variety $\mathcal{V}$ such that the congruence lattice are exactly the distributive algebraic lattices?

Let $\mathcal{W}$ be an arbitrary variety. The lattice-variety generated by all lattices $Con(K)$ $K \in \mathcal{W}$ is called congruence variety. About congruence varieties we refer to Bjarni Jõnson important lectures [4] hold on Vancouver Congress. The variety of all distributive lattices is trivially a congruence variety.

Problem. Does there exist for every congruence variety $\mathcal{U}$ a variety $\mathcal{D}$ such that every algebraic lattice $L \in \mathcal{U}$ is the congruence lattice of some algebra $A \in \mathcal{D}$ ?

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Math. Inst. of the
Hungarian Academy of Sciences
Budapest V.
Realtanoda U. 13-15
Ungarn