COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
14. LATTICE THEORY, SZEGED (HUNGARY), 1974.

LATTICES GENERATED BY PARTIAL LATTICES

E.T. SCHMIDT

Dedicated to L. Rédei on his 75-th birthday

1. INTRODUCTION

Let M be a lattice and D be a distributive lattice with 0 and 1. If a/b is a prime quotient of M then we can define a partial lattice $M(aDb)$ as follows: $M(aDb)$ is the set-theoretical union of M and D, and we identify a with 1 and b with 0; M and D are sublattices and $m \vee d$, $m \wedge d$ $(m \in M,\ d \in D)$ are defined only for $d \in M \cap D = \{a, b\}$. Shortly speaking we put the distributive lattice D into the prime quotient a/b of M. We will show that in every equational class $\mathscr{K}$ containing M there exists a lattice which contains a relative sublattice isomorphic to $M(aDb)$. In this paper we define a special lattice $M[D]$ (an extension of M by D) containing the relative sublattice $M(aDb)$. The construction is a generalization of that given by R. W. Quackenbush [7] for the case if M is a bounded distributive lattice. Let M_3 be the five-element modular, non-distributive lattice, $M_3[D]$ was defined in [9] and later discussed by A. Mitschke and R. Wille [5]. Let $\widetilde{M}(aDb)$ be the sublattice of $M[D]$ generated by $M(aDb)$. We prove that the quotient sublattice of a/b in $\widetilde{M}(aDb)$ (and in $M[D]$) is isomorphic to D, and $\Theta(a/b)/\omega$ in $\widetilde{M}(aDb)$

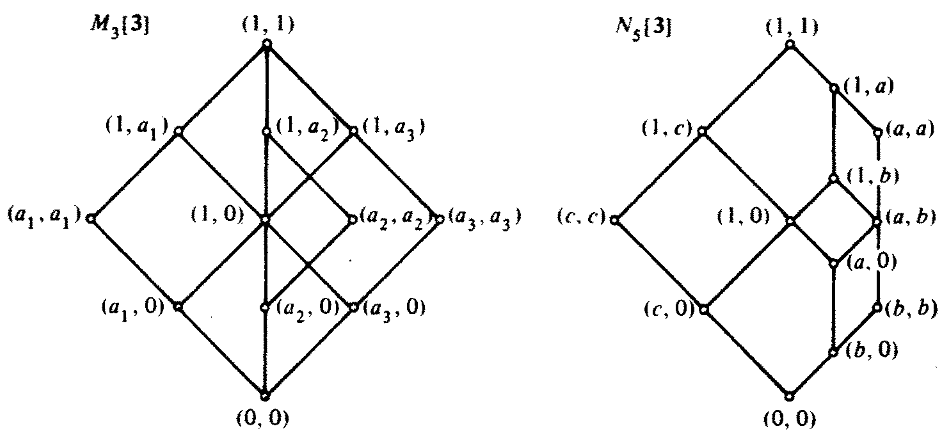
is isomorphic to $\Theta(D)$. For modular M, these results have been proved independently by R. Freese [3]. Finally we give a shorter proof of the main theorem of [5].

2. THE FINITE CASE

Let first D be a finite distributive lattice with the ultrafilters $Q_1, Q_2, \ldots, Q_n$. (By the definition of the ultrafilter $Q_i \neq D$).

For an arbitrary lattice M we define a special subdirect power $M[D]$ of M as follows: $M[D]$ contains all $d = (d_1, d_2, \ldots, d_n) \in M^n$ for which $Q_i \supset Q_j$ implies $d_i \geqslant d_j$. $M[D]$ is obviously a sublattice of M^n, hence a subdirect power of M. Let X be the poset of all ultrafilters of D. Then $M[D]$ *is the lattice of all monotone maps of* X *into the lattice* M.

Before we discuss some interesting properties of this lattice, let us take too examplars for $M[D]$. Let D be the three-element chain **3**. We shall consider $M = M_3$ (with the elements $0, a_1, a_2, a_3, 1$), and in the second case $M = N_5$, the five-element non-modular lattice (with the elements $0, 1, a, b, c$; $a > b$). The corresponding lattices $M_3[3]$ and $N_5[3]$ are:



If 2 is the two-element distributive lattice then $M[2] \cong M$. The correspondence

$$m \to f_m = (m, m, \ldots, m) \qquad (m \in M)$$

is the canonical embedding of M into M^n and by the definition of $M[D]$ every f_m belongs to $M[D]$. Let a/b a prime quotient of M. We shall show that the corresponding quotient f_a/f_b of $M[D]$ is isomorphic to D. But by the Birkhoff – Stone representation theorem we have for every $d \in D$ the correspondence $d \to g_d = (d_1, d_2, \ldots, d_n) \in 2^n$, where $d_i = 1$ if $d \in Q_i$ and $d_i = 0$ if $d \notin Q_i$. If $d_i \ngeq d_j$ then $d_i = 0$, $d_j = 1$ hence $d \notin Q_i$, $d \in Q_j$ and therefore $Q_i \not\supseteq Q_j$; hence $Q_i \supset Q_j$ implies $d_i \geqslant d_j$. We define $g_d(a/b) = (y_1, y_2, \ldots, y_n) \in f_a/f_b$, where $y_i = a$ if $d_i = 1$ and $y_i = b$ otherwise. Then $d \to g_d(a/b) \in M[D]$ is an isomorphism between D and f_a/f_b. The elements f_m $(m \in M)$ and y $(y \in f_a/f_b)$ form a relative sublattice isomorphic to $M(aDb)$, hence

Proposition 1. *$M(aDb)$ is isomorphic to a relative sublattice of $M[D]$.*

Let $\mathscr{K}$ be an equational class containing M (we assume that $|M| > 1$).

Proposition 2. *The free lattice $F_{\mathscr{K}}(M(aDb))$ over $\mathscr{K}$ generated by $M(aDb)$ exists.*

Let $\widetilde{M}(aDb)$ be the sublattice of $M[D]$ generated by $M(aDb)$. $M[D]$ is a subdirect power of M hence

Proposition 3. *Every congruence relation of $a/b \, (= f_a/f_b)$ can be extended to $\widetilde{M}(f_a/f_b) \, (= \widetilde{M}(aDb))$, hence in $\widetilde{M}(aDb)$ $\Theta(a, b)/\omega$ is isomorphic to $\Theta(D)$.*

Let M be a bounded lattice. Then the g_d defined above can be taken as an element of $M[D]$.

Proposition 4. *If M is a bounded lattice then $M[D]$ is generated by*

$$\{f_m \mid m \in M\} \cup \{g_d \mid d \in D\}.$$

Proof. (See [4]). Let $h = (h_1, h_2, \ldots, h_n) \in M[D]$. Let $X_{h_i} = \{Q_j \mid h_j \geqslant h_i\}$. Then X_{h_i} is an increasing subset of X (a subset E of X is increasing if $x \in E$, $y \geqslant x$ imply $y \in E$). Hence there exists a unique element $e_i \in D$ such that $X_{h_i} = \{Q_j \mid e_i \in Q_j\}$ (e_i is the minimal element of the intersection $\cap Q_j$ of all $Q_j \in X_{h_i}$). We prove that $h = \bigvee_{i=1}^{n} (f_{h_i} \wedge g_{e_i})$. Let us take:

$$\bigvee_{i=1}^{n} (f_{h_i} \wedge g_{e_i})(Q_j) = \vee \{h_i \mid e_i \in Q_j\} =$$

$$= \vee \{h_i \mid Q_j \in X_{e_i}\} = \vee \{h_i \mid h_j \geqslant h_i\} = h_j .$$

3. THE GENERAL DEFINITION OF $M[D]$

$\mathscr{D}$ will denote the equational class of bounded distributive lattices. If $D \in \mathscr{D}$, then the set X of all ultrafilters of D becomes a compact totally order disconnected space by identifying X with the set $\mathrm{Hom}_{\mathscr{D}}(D, 2)$ of homomorphism onto 2. Let M be an arbitrary lattice.

Definition. $\mathfrak{C}_{\leqslant}(X, M)$ is the lattice of all continuous monotone maps of the compact totally order disconnected space X into the discrete space M.

If D is a finite distributive lattice then by the definition of $M[D]$ in the previous paragraph we obtain that $M[D] = \mathfrak{C}_{\leqslant}(X, M)$. Therefore we define in the general case: $M[D]$ *is the lattice* $\mathfrak{C}_{\leqslant}(X, M)$.

Remark. We give a motivation for the definition of $M[D]$. Let M be a bounded *distributive* lattice. In [7] R.W. Quackenbush has defined $M[D]$ as follows: $M[D]$ is the subalgebra of M^X generated by $\{f_m \mid m \in M\} \cup \{g_d \mid d \in D\}$, where for all $Q \in X$ $f_m(Q) = m$ and

$$g_d(Q) = \begin{cases} 1 & \text{if } d \in Q, \\ 0 & \text{if } d \notin Q. \end{cases}$$

Brian A. Davey [2] has shown, that (for distributive M) $M[D]$ is isomorphic to $\mathfrak{C}_{\leqslant}(X, M)$. (The proof is essentially the same as the proof of Proposition 4). The proof does not use the distributivity of M, we have therefore

Proposition 5. *If M is a bounded lattice, then $M[D]$ is the sublattice of M^X generated by*

$$\{f_m \mid m \in M\} \cup \{g_d \mid d \in D\}.$$

The set $\{f_m \mid m \in M\}$ is a sublattice of $M[D]$ isomorphic to M. Let a/b be a prime quotient of M. Then $f_a/f_b \in M[D]$ is isomorphic to $\mathfrak{C}_{\leqslant}(X, 2)$. By a theorem of H.A. Priestley [6] this last lattice is isomorphic to D, hence we have the following

Theorem 1. *Let a/b be a prime quotient in a lattice M and let D be a bounded distributive lattice. Then there exists a lattice $M[D]$ containing the relative sublattice $M(aDb)$ such that the quotient a/b of $M[D]$ is isomorphic to D, and $\Theta(a, b)/\omega$ is isomorphic to $\Theta(D)$.*

For modular lattices this theorem was proved independently by R. Freese [3].

Corollary 1. *If $\mathscr{K}$ is an equational class containing M, then $F_{\mathscr{K}}(M(aDb))$ exists.*

Let $\widetilde{M}(aDb)$ denote the sublattice of $M[D]$ generated by $M(aDb)$ (more precisely by $M(f_a D f_b)$). Then $\widetilde{M}(aDb)$ has the following characterization

Corollary 2. *$\widetilde{M}(aDb)$ is the sublattice of $M[D]$ generated by*

$$\{f_m \mid m \in M\} \cup \{h_d \mid d \in D\}$$

where

$$h_d(Q) = \begin{cases} a & \text{if } d \in Q, \\ b & \text{if } d \notin Q. \end{cases}$$

4. FINITE MODULAR LATTICES

If M is a simple modular lattice then obviously $\widetilde{M}(aDb) = M[D]$. The lattice given by the first diagram is also $\widetilde{M}_3(a_1 D0)$. Another characterization for $M_3[D]$ was given in [8], [9]: let L be the poset of all triples (x, y, z) $(x, y, z \in D)$ with the property $x \wedge y = y \wedge z = x \wedge z$, ordered by the rule: $(x, y, z) \leqslant (x', y', z')$ iff $x \leqslant x'$, $y \leqslant y'$, $z \leqslant z'$. The lattice operations of L are:

$$(x_1, y_1, z_1) \wedge (x_2, y_2, z_2) = (x_1 \wedge x_2,\ y_1 \wedge y_2,\ z_1 \wedge z_2)$$

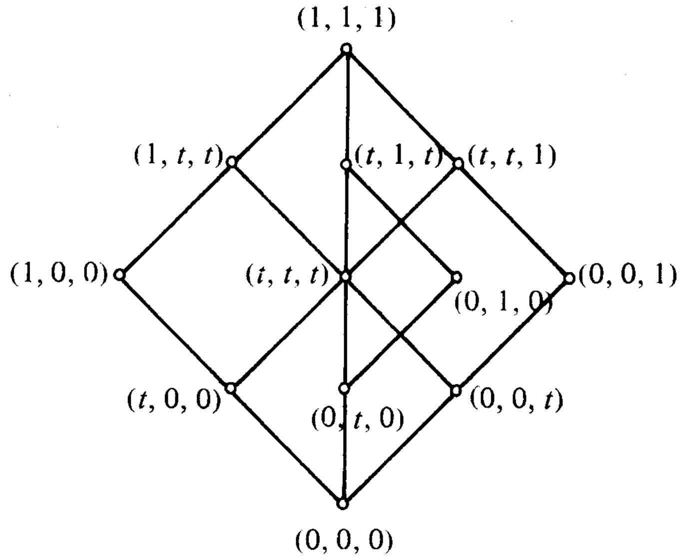
and

$$\begin{aligned}(x_1, y_1, z_1) \vee (x_2, y_2, z_2) = (&(x_1 \vee x_2) \vee [(y_1 \vee y_2) \wedge (z_1 \vee z_2)],\\ &(y_1 \vee y_2) \wedge [(x_1 \vee x_2) \wedge (z_1 \vee z_2)],\\ &(z_1 \vee z_2) \vee [(x_1 \vee x_2) \wedge (y_1 \vee y_2)]).\end{aligned}$$

Let a_1, a_2, a_3 denote the atoms of M_3, then the injections $a_1 \to$ $\to (1, 0, 0)$. $a_2 \to (0, 1, 0)$, $a_3 \to (0, 0, 1)$, $d \to (d, 0, 0)$ $(d \in D)$ defines an embedding of $M(a_1 D0)$ into L. In [8], [9] it was proved that the congruence lattices of D and L are isomorphic, moreover every congruence relation of D can be extended to L. This yields that L is a subdirect power of M_3. Let P_i be a ultrafilter of D, then we denote by $\bar{\Theta}[P_i]$ the extension of the congruence relation $\Theta[P_i]$ to L. Let u be an element of L, then we can take the mapping $\hat{u}$: $X \to M_3$ where X is the set of ultrafilters of D for which $\hat{u}(P_i)$ is the image of a_1 by the natural homomorphism φ: $L \to L/\bar{\Theta}[P_i]$ $(L/\bar{\Theta}[P_i]$ is isomorphic to $M_3)$. We get

Proposition 6. *L is isomorphic to* $M_3[D]$.

The given representation of $M_3[3]$ is shown by the next diagram (the elements of 3 are $0, t, 1$).



Problem. Is it possible to give a similar characterization for $M[D]$ if M is a finite simple complemented modular lattice?

Let $p_1, p_2, \ldots, p_n$ be the atoms of M. An element $(x_1, x_2, \ldots, x_n)$ of D^n is called normal if $p_i \vee p_j \geq p_k$ implies $x_i \wedge x_j = x_i \wedge x_k = x_j \wedge x_k$. Conjecture: $M[D]$ is the poset of all normal elements.

5. THE CHARACTERISATION OF $F_{\mathcal{M}}(M_3(0Da_1))$.

($\mathcal{M}$ denotes the equational class of modular lattices.) In this section we give a simple proof for the main theorem of [5]. The proof is based on an interesting property of M_3.

Proposition 7. *Let M_3 be a sublattice of a modular lattice L. If $f(x)$ and $g(x)$ are unary algebraic functions over M_3 then $f(0) = g(0)$ and $f(a_1) = g(a_1)$ imply $f(x) = g(x)$ for every $x \in L$ $(x \in a_1/0)$.*

Proof. The product of two unary algebraic functions f_1 and f_2 is defined by $f_1 f_2(x) = f_1(f_2(x))$. Let us take the following special unary algebraic functions over M_3. $f_i = x \vee a_i$, $g_i(x) = x \wedge a_i$, $i(x) = x$ $(= x \vee 0 = = x \wedge 1)$. Let f be a uary algebraic function such that $f(0) \neq f(a_1)$.

Then f is obviously the product of these special functions. Let x be an element of $a_1/0$. Then for $u, v \in \{a_1, a_2, a_3\}$, $u \neq v$ we have:

(1) $\quad f_u g_v f_u(x) = f_u(x)$ and $g_v f_u g_v(x) = g_v(x)$.

If u, v, w are three distinct elements of $\{a_1, a_2, a_3\}$ then we prove:

(2) $\quad f_u g_v f_w = f_u g_w f_v$ and $g_u f_v g_w = g_u f_w g_v$.

Take $f_u g_v f_w(x) \vee f_u g_w f_v(x)$. We can assume that $x \leqslant a_1$, since for other x (2) is obviously satisfied. By the modularity we get

$$f_u g_v f_w(x) \vee f_u g_w f_v(x) = \{[x \vee w) \wedge v] \vee u\} \vee \{[x \vee v) \wedge w] \vee u\} =$$

$$= \{(x \vee w) \wedge [v \vee ((x \vee v) \wedge w)]\} \vee u = [(x \vee w) \wedge (x \vee v)] \vee u =$$

$$= x \vee (w \wedge (x \vee v)) \vee u = [(x \vee v) \wedge w] \vee (u \vee x) =$$

$$= [(x \vee v) \wedge w] \vee u = f_v g_w f_u(x).$$

By the symmetry of v and w we get (2). Using (1) and (2) a simple discussion proves our lemma.

Theorem 2 (A. Mitschke and R. Wille [5]). *Let N be a modular lattice and let $M_3(0Da_1)$ be a relative sublattice of N. The following statements are equivalent:*

(a) *N is generated by $M_3(0Da_1)$;*

(b) *N is isomorphic to $F_{\mathcal{M}}(M_3(0Da_1))$;*

(c) *N is isomorphic to the subdirect power of M_3 containing all quasi-real, continuous mappings of the Stone space $S(D)$ into the T_0-space M_3 with the subbasis $\{[x) \mid x \in M_3\}$. (For the notion see [4]).*

Proof of Theorem 1. Let $M_3(0Da_1)$ be a relative sublattice of the modular lattice L, and denote by N the sublattice generated by $M_3(0Da_1)$. We prove that N is isomorphic to N_0 where N_0 is the lattice obtained from D by taking all (x, y, z) $(x, y, z \in D)$ with $x \wedge y = x \wedge z = y \wedge z$. Put $D' = \{x \in L \mid x = (d \vee a_3) \wedge a_2) \vee a_1,\ d \in D\}$. Then D' is a distributive sublattice of $a_1 \vee a_2/a_1$ isomorphic to D.

(i) For $x \in N$ we set $x_1 = x \wedge a_1$, $x_2 = ((x \wedge a_2) \vee a_3) \wedge a_1$, $x_3 = ((x \wedge a_3) \vee a_2) \wedge a_1$. By the modularity of N we have $x_1 \wedge x_2 = (x \wedge a_1) \wedge ((x \wedge a_2) \vee a_3) \wedge a_1 = [x_1 \wedge ((x \wedge a_2) \vee a_3)] \wedge a_1 = [(x \wedge a_2) \vee (x \wedge a_3)] \wedge a_1$. By the symmetry of a_2 and a_3 we get $x_1 \wedge x_3 = x_1 \wedge x_2$. Finally, $x_2 \wedge x_3 = [((x \wedge a_2) \vee a_3) \wedge a_1] \wedge [((x \wedge a_3) \vee a_2) \wedge a_1] = [(x \wedge a_2) \vee a_3] \wedge [(x \wedge a_3) \vee a_2] \wedge a_1 = \{(x \wedge a_2) \vee [a_3 \wedge ((x \wedge a_3) \vee a_2)]\} \wedge a_1 = [(x \wedge a_2) \vee (x \wedge a_3)] \wedge a_1$. Thus $x_1 \wedge x_2 = x_1 \wedge x_3 = x_2 \wedge x_3$, hence using the distributivity of D:

$$\text{if} \quad x_1, x_2, x_3 \in D \quad \text{then} \quad x_1 = (x_1 \vee x_2) \wedge (x_1 \vee x_3). \tag{3}$$

(ii) Put $x^{(1)} = a_1 \vee (x \wedge a_2) \vee (x \wedge a_3)$, $x^{(2)} = a_2 \vee (x \wedge a_1) \vee (x \wedge a_3)$, $x^{(3)} = a_3 \vee (x \wedge a_1) \vee (x \wedge a_2)$.

From (1) we get

$$\begin{aligned} & a_1 \wedge x^{(2)} \wedge x^{(3)} = (a_1 \wedge x^{(2)}) \wedge (a_1 \wedge x^{(3)}) = \\ & = [a_1 \wedge (a_2 \vee (x \wedge a_1) \vee (x \wedge a_3))] \wedge [a_1 \wedge (a_3 \vee (x \wedge a_1) \vee \\ & \vee (x \wedge a_2))] = [(x \wedge a_1) \vee (a_1 \wedge (a_2 \vee (x \wedge a_3))] \wedge [(x \wedge a_1) \vee \\ & \vee (a_1 \wedge (a_3 \vee (x \wedge a_2))] = (x_1 \vee x_3) \wedge (x_1 \vee x_2) = x_1. \end{aligned} \tag{4}$$

Obviously $(x \wedge a_2) \vee (x \wedge a_3) \leqslant x^{(2)}$ and $x^{(3)}$, i.e. $(x \wedge a_2) \vee (x \wedge a_3) \leqslant x^{(2)} \wedge x^{(3)}$. Applying (2), from these inequalities, we get $x^{(1)} \wedge x^{(2)} \wedge x^{(3)} = [a_1 \vee (x \wedge a_2) \vee (x \wedge a_3)] \wedge (x^{(2)} \wedge x^{(3)}) = (a_1 \wedge x^{(2)} \wedge x^{(3)}) \vee (x \wedge a_2) \vee (x \wedge a_2) \vee (x \wedge a_3) = x_1 \vee (x \wedge a_2) \vee (x \wedge a_3) = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3)$. Thus

$$\begin{aligned} & \text{If} \quad x \in N \quad \text{and} \quad x_i \in D \quad (i = 1, 2, 3) \\ & \text{then} \quad x = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3) \\ & \text{implies} \quad x = (x \vee a_1) \wedge (x \vee a_2) \wedge (x \vee a_3). \end{aligned} \tag{5}$$

(iii) Let A be the set $\{x;\ x = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3),\ x_1, x_2, x_3 \in D\}$. If $x, y \in A$ then $((x \vee y) \wedge a_1) \vee ((x \vee y) \wedge a_2) \vee ((x \vee y) \wedge a_3) = x \vee y$, i.e. A is a join semilattice.

(iv) By Proposition 7 if $x \in A$ then

$x \vee a_1 = (x \wedge a_2) \vee (x \wedge a_3) \vee a_1$, $((x \vee a_2) \wedge a_3) \vee a_1$, $((x \vee a_3) \wedge a_2) \vee a_1$ are in D' hence from (5) it follows that

$$
(5') \quad \begin{aligned}
&\text{If } \; x \in N \; \text{ and} \\
&x \vee a_1, ((x \vee a_2) \wedge a_3) \vee a_1, ((x \vee a_3) \wedge a_2) \vee a_1 \in D' \\
&\text{then } \; x = (x \vee a_1) \wedge (x \vee a_2) \wedge (x \vee a_3) \\
&\text{implies } \; x = (x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3).
\end{aligned}
$$

A is therefore a meet semilattice too. A contains obviously the relative sublattice $M_3(0Da_1)$. The representation $x = x_1 \vee ((x_2 \vee a_3) \wedge a_2) \vee \vee ((x_3 \vee a_2) \wedge a_3)$ implies that A is generated by this partial lattice, hence A and N are isomorphic.

(v) Finally we prove that N and N_0 are isomorphic too. Let us take the correspondence $x \to (x_1, x_2, x_3)$. But $x_1 \wedge x_2 = x_1 \wedge x_3 = x_2 \wedge x_3$, i.e. $(x_1, x_2, x_3) \in N_0$. Conversely let $(u, v, w) \in N_0$ and set $x = u \vee [(v \vee a_3) \wedge a_2] \vee [(w \vee a_2) \wedge a_3]$. It is easy to verify that $x_1 = u$, $x_2 = v$, and $x_3 = w$, the given correspondence is a one-to-one order preserving mapping. Thus $N \cong N_0$. The conditions (a) and (b) are also equivalent. For the proof of the equivalence of (a) and (c) we refer to [5].

Let L be a lattice from the equational class $\mathscr{K}$, A prime quotient a/b of L is called $\mathscr{K}$-pure if for every extension $M \in \mathscr{K}$ of L and for any two unary algebraic functions $f(x)$, $g(x)$ over L the conditions $f(a) = g(a)$, $f(b) = g(b)$ imply $f(x) = g(x)$ for every $x \in a/b$, $x \in M$. The finite lattice L is $\mathscr{K}$-pure if every prime quotient is $\mathscr{K}$-pure. By Proposition 7 M_3 is $\mathscr{M}$-pure.

Problem. Is it true that $\widetilde{M}(aDb) \cong F_{\mathscr{M}}(aDb)$ for $M \in \mathscr{M}$ if and only if M is $\mathscr{M}$-pure?

REFERENCES

[1] R. Balbes – Ph. Dwinger, Coproducts of boolean algebras and chains with applications to Post algebras, *Coll. Math.*, 23 (1971), 15-25.

[2] B.A. Davey, Free products of bounded distributive lattices, *Algebra Universalis*, 4 (1974), 106-107.

[3] R. Freese, Congruence lattices of finitely generated modular lattices, Preprint.

[4] K.H. Hoffmann – K. Keimel, A general character theory for partially ordered sets and lattices, *Memoires Amer. Math. Soc.*, 122 (1972), 119.

[5] A. Mitschke – R. Wille, Freie modulare Verbände $FM({}_D M_3)$, *Proc. Univ. of Houston*, (1973), 383-396.

[6] H.A. Priestley, Ordered topological spaces and the representation of distributive lattices, *Proc. London Math. Soc.*, 24 (1972), 507-530.

[7] R.W. Quackenbush, Free products of bounded distributive lattices, *Algebra Universalis*, 2 (1972), 393-394.

[8] E.T. Schmidt, *Kongruenzrelationen algebraischer Strukturen*, Math. Forschungsber. XXV Berlin (1969).

[9] E.T. Schmidt, Zur Charakterisierung der Kongruenzverbände der Verbände, *Mat.-Fyz. Časopis Slovensk. Akad. Vied.*, 18 (1968), 3-20.

E.T. Schmidt

Mathematical Institute of the Hungarian Academy of Sciences, 1053 Budapest, Reáltanoda u. 13-15.