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Standard sublattices

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0. Introduction

The concept of a standard ideal of a lattice was introduced in [1]. An ideal S of a lattice L is called standard if

$$I \wedge (S \vee K) = (I \wedge S) \vee (I \wedge K) \tag{1}$$

holds for any pair of ideals I, K of L, where $\vee$ and $\wedge$ denote the lattice-operations of the ideal-lattice $I(L)$ of L.

This concept is a generalization of neutral ideals and has many useful properties. Standard ideals play the same role for lattices as invariant subgroups for groups. A congruence of a group is determined by any congruence-class. However, even this does not hold for congruences generated by standard ideals. So, we should take into consideration all 'standardlike' possible congruenceclasses.

The aim of this paper is to give a generalization of standard ideals for convex sublattices, called standard sublattices, and to prove that many important properties of standard ideals are also valid for standard sublattices.

1. The definition of a standard sublattice

We shall denote by $\cup$ and $\cap$ the set-theoretical and by $\vee$ and $\wedge$ the lattice-theoretical operations. $\emptyset$ denotes the empty set. The convex sublattice generated by a subset A of the lattice L will be denoted by $\langle A \rangle$. Let A and B be two (nonempty) subsets of the lattice L. Then we define

$$A \vee B = \langle \{a \vee b /\ a \in A,\ b \in B\} \rangle$$
$$A \wedge B = \langle \{a \wedge b /\ a \in A,\ b \in B\} \rangle,$$

i.e., $A \vee B$ and $A \wedge B$ are the convex sublattices of L generated by the elements $a \vee b$ and $a \wedge b$ ($a \in A$, $b \in B$), respectively.

Let us remark, if A and B are both ideals (or both dual-ideals) then $A \vee B$ and $A \wedge B$ are exactly the join and the meet of A and B in the ideal-lattice. However, in the general case neither $A \leqq A \vee B$ nor $A \wedge B \leqq A$ are valid. For example, if $A = \{a\}$ and $B = \{b\}$ then both inequalities imply $A = B$.

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DEFINITION. A convex sublattice S of a lattice L is called a standard sublattice if

$$I \wedge \langle S, K\rangle = \langle I \wedge S, I \wedge K\rangle \tag{2}$$

and

$$I \vee \langle S, K\rangle = \langle I \vee S, I \vee K\rangle \tag{3}$$

hold for any pair $\{I, K\}$ of convex sublattices of L, whenever neither $S \cap K$ nor $I \cap \langle S, K\rangle$ are empty. (Thus, the word 'standard' implies convexity.)

PROPOSITION 1. *For each $s \in L$, $\{s\}$ is a standard sublattice of L.*

Proof. $\{s\} \cap K \neq \emptyset$ implies $s \in K$, yielding $\langle \{s\}, K\rangle = K$, $I \wedge \{s\} \leqq I \wedge K$, and $I \vee \{s\} \leqq I \vee K$. Thus, in (2) and (3) both the left and the right hand sides of the equations are $I \wedge K$ and $I \vee K$, respectively.

Now, we are going to prove that we have indeed a generalization. To do this we need the following

LEMMA 1. *For the convex sublattices A and B of the lattice L the equalities*

$$A \wedge (B] = (A] \wedge (B] \quad \textit{and} \quad \langle A, (B]\rangle = (A] \vee (B]$$

hold, where $(X]$ denotes the ideal generated by X.

Proof. $A \leqq (A]$ implies, obviously, both $A \wedge (B] \leqq (A] \wedge (B]$ and $\langle A, (B]\rangle \leqq \langle (A], (B]\rangle = (A] \vee (B]$.

(i) $x \in (A] \wedge (B]$ implies $x \in (A]$ and $x \in (B]$, for $(A] \wedge (B] = (A] \cap (B]$. Since the ideal $(A]$ consists of all elements of L having an upper bound in A we have $x \leqq a$ for some $a \in A$. Hence $x = a \wedge x \in A \wedge (B]$.

(ii) Let $x \in (A]$, i.e., $x \leqq a$ for some $a \in A$, and $y \in (B]$. Then we have, using the convexity of $\langle A, (B]\rangle$, by $y \leqq x \vee y \leqq a \vee y$ that $x \vee y \in \langle A, (B]\rangle$. Thus, $(A] \vee (B]$, the smallest convex sublattice containing all $x \vee y$ $(x \in (A], y \in (B])$, is contained in $\langle A, (B]\rangle$.

PROPOSITION 2. *An ideal S of a lattice L is standard if and only if it is a standard sublattice.*

Proof. Let us assume, first, that the ideal S is a standard sublattice of L. Then the ideals I and K are, of course, convex sublattices. Moreover $S \cap K \neq \emptyset$ and $I \cap \langle S, K\rangle \neq \emptyset$ are, clearly, satisfied. Thus we have by (2) and (3)

$$I \wedge \langle S, K\rangle = \langle I \wedge S, I \wedge K\rangle \quad \text{and} \quad I \vee \langle S, K\rangle = \langle I \vee S, I \vee K\rangle.$$

$\langle A, B\rangle = A \vee B$ for the ideals A and B, i.e., we arrive at

$$I \wedge (S \vee K) = (I \wedge S) \vee (I \wedge K) \quad \text{and} \quad I \vee (S \wedge K) = (I \vee S) \wedge (I \vee K).$$

The first equality gives us, precisely, that S is a standard ideal. (The second equality is an obvious one.)

Let, conversely, S be a standard ideal.

Using the obvious equality $(I \wedge K] = (I] \wedge (K]$ valid for any subset I and K of L, we have for the convex sublattice I and K of L, by Lemma 1:

$$I \wedge \langle S, K \rangle = I \wedge (S \vee (K]) = (I] \wedge (S \vee (K])$$

and

$$\langle I \wedge S, I \wedge K \rangle = \langle (I] \wedge S, I \wedge K \rangle = ((I] \wedge S) \vee (I \wedge K) = ((I] \wedge S) \vee ((I] \wedge (K]).$$

The standard ideal property for S yields (2).

We claim, now, that (3) is valid for every ideal S of L.

$S \leqq \langle S, K \rangle$ and $K \leqq \langle S, K \rangle$ imply $\langle I \vee S, I \vee K \rangle \leqq I \vee \langle S, K \rangle$. By Lemma 1 $\langle S, K \rangle = S \vee (K]$, i.e., $I \vee \langle S, K \rangle$ is, clearly, the convex sublattice generated by the elements of the form $x \vee (s \wedge y)$ where $x \in I$, $s \in S$, $y \leqq t$ for some $t \in K$. By convexity

$$x \vee s,\ (x \vee s) \wedge (x \vee t) \in \langle I \vee S, I \vee K \rangle$$

and

$$x \vee s \leqq x \vee (s \vee y) \leqq (x \vee s) \vee (x \vee t) \quad \text{imply} \quad x \vee (s \vee t) \in \langle I \vee S, I \vee K \rangle.$$

Using the convexity of $\langle I \vee S, I \vee K \rangle$ again we have $I \vee \langle S, K \rangle \leqq \langle I \vee S, I \vee K \rangle$ which finishes the proof.

Now, we prove that standard sublatices have similar characterisations to those of standard ideals in [1].

THEOREM 1. *The following four conditions are equivalent for each convex sublattice S of a lattice L.*

(α) *S is a standard sublattice.*

(β) *Let $K \leqq L$ be any convex sublattice of L such that $K \cap S \neq \emptyset$. Then, to each $x \in \langle S, K \rangle$ there exist $s_1, s_2 \in S$, $a_1, a_2 \in K$ such that:*

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2).$$

(β') *Let K be as before. Then, for each S and to each elements $x \in \langle S, K \rangle$ and to each $s_2, s_1' \in S$ there are elements $s_1, s_2' \in S$, $a_1, a_2 \in K$ such that:*

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2)) = (x \vee s_2') \wedge (x \vee (a_2 \wedge s_1')).$$

(γ) *The relation $\theta[S]$ on L defined by:*

'$x \equiv y(\theta[S])$ *if and only if*

$$x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y) \quad \text{and} \quad x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$$

with suitable t, s *in* S' *is a congruence relation.*

Proof. We will prove the equivalence of the four conditions cyclically.

(α) implies (β). Let $K \cap S \neq \emptyset$ and let $x \in \langle S, K\rangle$. For $I = \{x\}$ the relation $I \cap \langle S, K\rangle$ is satisfied. x is, clearly, the greatest element of $I \wedge \langle S, K\rangle$, i.e., by (2) the greatest element of $\langle I \wedge S, I \wedge K\rangle$. Thus, $x = (x \wedge s_1) \vee (x \wedge a_1)$ with suitable $s_1 \in S$, $a_1 \in K$.

The dual property follows, similarly, from (3).

(β) implies (β'). Let $K' = \langle\{s\}, K\rangle$ where s is an arbitrary element of S. Then $s \in K'$ implies $S \cap K' \neq \emptyset$. Further, $K \leqq K' \leqq \langle S, K\rangle$ yields $\langle S, K\rangle = \langle S, K'\rangle$. Thus, for any $x \in \langle S, K\rangle$ the equality

$$x = (x \wedge s_1) \vee (x \wedge \bar{a}_1) \qquad (s_1 \in S, \bar{a}_1 \in K')$$

holds, by (β). However $\bar{a}_1 \leqq a_1 \vee s$ $(a_1 \in K)$ implies $x \leqq (x \wedge s_1) \vee (x \wedge (a_1 \vee s)) \leqq x \vee x = x$, i.e.,

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s)). \tag{4}$$

We have dually, for an arbitrary $s' \in S$ the equality

$$x = (x \vee s_2) \wedge (x \vee (a_2 \wedge s')) \qquad (s_2 \in S, a_2 \in K). \tag{5}$$

Remark. Substituting in (β') s_1, s_2', a_1, a_2 by s_3, s_4', a_3, a_4, respectively, where $s_3 \geqq s_1$, $s_4' \leqq s_2'$, $a_3 \geqq a_1$, $a_4 \leqq a_2$ $(s_3, s_4' \in S, a_3, a_4 \in K)$ we also get equality.

Proof. It is enough, by duality, to deal only with s_3 and a_3. Monotonity implies:

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2) \leqq (x \wedge s_3) \vee (x \wedge (a_3 \vee s_2)) \leqq x \vee x = x,$$

proving the statement.

(β') implies (γ). Let $\theta[S]$ be defined as follows: '$x \equiv y(\theta[S])$, for $x \geqq y$, if and only if $y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ hold for suitable s, t in S'.

Let us mention, that we may choose s and t such that $s \geqq t$ holds, because of the monotonity. It is not too hard to verify that $\theta[S]$ is an equivalence relation (see [2] p. 24.). We shall prove, that $x \wedge z \equiv y \wedge z(\theta[S])$ and $x \vee z \equiv y \vee z(\theta[S])$ are also valid for every $z \in L$. Since the definition of $\theta[S]$ is self-dual it is enough to prove only the first statement.

$y \leqq x$ implies $y \wedge z \leqq x \wedge z$ thus the trivial inequality $y \wedge z \leqq t \vee (y \wedge z)$ gives us $y \wedge z \leqq (t \vee (y \wedge z)) \wedge (x \wedge z)$. On the other hand

$$(t \vee (y \wedge z)) \wedge (x \wedge z) \leqq (t \vee y) \wedge x \wedge z = ((t \vee y) \wedge x) \wedge z = y \wedge z,$$

i.e.,

$$y \wedge z = ((y \wedge z) \vee t) \wedge (x \wedge z).$$

Now, let K be the convex sublattice $\langle t \wedge y \wedge z, y \rangle$. We have $s, t \wedge y \wedge z \in \langle S, K \rangle$, for $s \in S$, $t \wedge y \wedge z \in K$. By the convexity of $\langle S, K \rangle$ the inequalities

$$t \wedge y \wedge z \leqq t \wedge y \leqq t \wedge x \leqq s \wedge x \leqq s$$

imply $s \wedge x \in \langle S, K \rangle$ yielding $x = (s \wedge x) \vee y \in \langle S, K \rangle$. Thus, the convexity of $\langle S, K \rangle$ and the inequalities

$$t \wedge y \wedge z \leqq y \wedge z \leqq x \wedge z \leqq x$$

imply $y \wedge z$, $x \wedge z \in \langle S, K \rangle$.

Since $t \in S$, we have, by (β'), elements $s' \in S$, $a_1 \in K$ such that

$$x \wedge z = ((x \wedge z) \wedge s') \vee (x \wedge z \wedge (a_1 \vee t)).$$

As y is the greatest element of K, we obtain, by the remark, that

$$x \wedge z = ((x \wedge z) \wedge s') \vee ((x \wedge z) \wedge (y \vee t)) = ((x \wedge z) \wedge s') \vee ((y \vee t) \wedge x) \wedge z) = ((x \wedge z) \wedge s') \vee (y \wedge z).$$

Hence, $\theta[S]$ is a congruence relation.

(γ) implies (α). It is enough to prove (2). $S, K \leqq \langle S, K \rangle$ implies $\langle I \wedge S, I \wedge K \rangle \leqq I \wedge \langle S, K \rangle$, i.e. we have to prove $I \wedge \langle S, K \rangle \leqq \langle I \wedge S, I \wedge K \rangle$. First we prove that each $u \in I \cap \langle S, K \rangle$ is contained in $\langle I \wedge S, I \wedge K \rangle$. Let v be an element of the nonempty set $S \cap K$. $\langle S, K \rangle$ is, obviously, the set of all elements y with $s_1 \wedge k_1 \leqq y \leqq s_2 \vee k_2$ ($s_1, s_2 \in S$, $k_1, k_2 \in K$). Moreover, monotonity implies that we may also suppose $s_1 \leqq v \leqq s_2$ and $k_1 \leqq v \leqq k_2$. The same hold for u, for it belongs to $\langle S, K \rangle$. $s_1 \equiv s_2 (\theta[S])$ implies $k_2 \vee s_2 \equiv k_2 \vee s_1 = k_2 (\theta[S])$. Then, there exists by (γ) an $s \in S$ such that $u = (u \wedge s) \vee (u \wedge k_2)$.

$u \in I$ implies $u = (u \wedge s) \vee (u \wedge k_2) \in \langle I \wedge S, I \wedge K \rangle$. Since $I \wedge \langle S, K \rangle$ is the smallest convex sublattice containing all elements of the form $i \wedge y$ ($i \in I$, $y \in \langle S, K \rangle$), it is enough to prove that all of these elements are in $\langle I \wedge S, I \wedge K \rangle$. Since $I \cap \langle S, K \rangle \neq \emptyset$ it contains an element u. We may choose $s_1 \in S$, $k_1 \in K$ such that $s_1 \wedge k_1 \leqq y$ as we have seen. Thus:

$$i \wedge (s_1 \wedge k_1) \leqq i \wedge y \leqq (i \vee u) \wedge (y \vee u).$$

It is enough to prove, by convexity, that

$$i \wedge (s_1 \wedge k_1) \in \langle I \wedge S, I \wedge K \rangle \quad \text{and} \quad i \wedge y \in \langle I \wedge S, I \wedge K \rangle \quad \text{for} \quad u \leqq i, u \leqq y,$$

since

$$u \in I \cap \langle S, K \rangle \quad \text{implies} \quad i \vee u \in I, y \vee u \in \langle S, K \rangle.$$

$i \wedge (s_1 \wedge k_1) = (i \wedge s_1) \wedge (i \wedge k_1)$ proves that this element belongs to $\langle I \wedge S, I \wedge K \rangle$.

$u \leqq i \wedge y \leqq i$ and $u \leqq i \wedge y \leqq y$ imply that $i \wedge y \in I \cap \langle S, K \rangle$, i.e., $i \wedge y$ is an element of $\langle I \wedge S, I \wedge K \rangle$. Hence, the theorem is proven.

COROLLARY 1. *If S is a standard sublattice than S is a congruence class by the congruence relation $\theta[S]$.*

Proof. Let $x \equiv y(\theta[S])$, $x > y$. We have to prove that if one of these elements belongs to S then both of them are in S. By the self-dual definition of standard sublattice we may assume $y \in S$. By condition (γ) $y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ with suitable $s, t \in S$. Then $x = (x \wedge s) \vee y \leqq (x \wedge (y \vee s)) \vee y \leqq x$, i.e., $x = (x \wedge (y \vee s)) \vee y$. Trivially $y \leqq x \wedge (y \vee s) \leqq y \vee s$ and $y, y \vee s \in S$. Hence, by the convexity of S, $x \wedge (y \vee s) \in S$ yielding $x = (x \wedge (y \vee s)) \vee y \in S$.

Let S be a standard sublattice of the lattice L. Then L/S denotes the lattice $L/\theta[S]$.

COROLLARY 2. *Let S and T be two standard sublattices. Then $S \cap T$ is either a standard sublattice or it is empty.*

Proof. We may assume that $S \cap T$ contains an element u. Let us suppose that $x \equiv y(\theta[S])$ and $x \equiv y(\theta[T])$ where $x > y$. Then we have $x = (s_1 \wedge x) \vee y$ with a suitable $s_1 \in S$ which may be supposed to be greater then u. On the other hand $x \equiv y(\theta[T])$ implies $s_1 \wedge x \equiv s_1 \wedge y(\theta[T])$ where $s_1 \wedge x \geqq s_1 \wedge y$ by the monotonity. Hence, (γ) implies $s_1 \wedge x = (t_1 \wedge (s_1 \wedge x)) \vee (s_1 \wedge y)$ with a $t_1 \geqq u$ in T. Consequently $x = (s_1 \wedge x) \vee y = [(t_1 \wedge (s_1 \wedge x)) \vee (s_1 \wedge y)] \vee y = ((t_1 \wedge s_1) \wedge x) \vee y$. But $t_1, s_1 \geqq u$ yield $t_1 \wedge s_1 \geqq u$, i.e., $t_1 \wedge s_1 \in T \cap S$. The duality finishes the proof.

COROLLARY 3. *The meet of a standard ideal and a standard dual ideal is a standard sublattice.*

Proof. By Proposition 2 and by the duality all standard ideals and standard dual ideals are standard sublattices. Corollary 2 completes the proof.

Remark. The converse of Corollary 3 is not true. For example in N_5 there are one-element subsets which are not the meet of a standard ideal and of a standard dual ideal. Proposition 1 proves our statement. We can prove more. We define on the set $\{a_0, \ldots, a_n, \ldots; b_0, \ldots, b_n, \ldots; c_0, \ldots, c_n, \ldots\}$ the following partial order:

$$a_0 < b_0, \qquad a_{i+1} < a_i, \qquad b_i < b_{i+1}, \qquad a_{i+1} < c_i < b_{i+1}.$$

It is easy to see that we have a subdirectly irreducible lattice, where $\theta(a_0, b_0)$ is the smallest congruence. This lattice has neither 0 nor 1, i.e., for each standard ideal or standard dual ideal the congruence class containing a_0 must also contain b_0. Thus the standard sublattice $\{a_0\}$ is not even the meet of two congruence-classes generated by a standard ideal and by a dual standard ideal.

2. Properties of standard sublattices

Firstly, we prove two Lemmas.

LEMMA 2. *Let S be a standard sublattice and I be an arbitrary convex sublattice of the lattice L such that $I \cap S \neq \emptyset$. Then $I \cap S$ is a standard sublattice of the lattice I.*

Proof. $I \cap S$ is obviously a convex sublattice of I. To prove that $I \cap S$ is standard we use condition (β). Each convex sublattice K of I is, clearly, a convex sublattice of L itself. Thus, by (β) each $x \in \langle S \cap I, K \rangle$ is to be written in the form

$$x = (x \wedge s) \vee (x \wedge a) \quad (s \in S, a \in K),$$

since $K \cap S = (K \cap I) \cap S = K \cap (I \cap S)$ is not empty.

We may assume, by monotonity, that both $s \geqq u$ and $a \geqq u$, where u is a given element of $K \cap (S \cap I)$. Then we have for $s' = (x \vee u) \wedge s$:

$$u = (x \vee u) \wedge u \leqq s'; \quad s' \leqq x \vee u; \quad s' \leqq s.$$

$u \in S \cap I$, $x \vee u \in I$, $s' \in S$ imply $s' \in S \cap I$. Hence, by $x \wedge s' = x \wedge (x \vee u) \wedge s = x \wedge s$,

$$x = (x \wedge s') \vee (x \wedge a) \quad (s' \in S \cap I, a \in K)$$

yields (β) in I. The duality finishes the proof.

LEMMA 3. *Let $x \to x'$ be a homomorphism of L onto L' and let S be a standard sublattice of L. The homomorphic image S' of S is a standard sublattice of L'.*

Proof. We shall prove (β) for S'. The coimage K of an arbitrary convex sublattice K' of L' is, obviously, a convex sublattice of L. $K' \cap S' \neq \emptyset$ implies $K \cap S \neq \emptyset$. Each $y' \in \langle S', K' \rangle$ has, clearly, a coimage $x \in \langle S, K \rangle$ for which, by (β)

$$x = (x \wedge s) \vee (x \wedge a) \quad (s \in S, a \in K)$$

holds. Then, $x' = y'$, $s' \in S'$, $a' \in K'$ proves the first statement of (β) for S'. The proof is completed by duality.

THEOREM 2. (The first isomorphism theorem). *Let L be a lattice, S a standard* *sublattice and I a convex sublattice of L such that $S \cap I \neq \emptyset$. Then $S \cap I$ is a standard sublattice of I and:*

$$\langle I, S\rangle/S \cong I/(I \cap S).$$

Proof. The first statement was proved in Lemma 2. Using the first isomorphism theorem for universal algebras it remains to prove that every congruence class of $\langle I, S\rangle$ may be represented by an element of I. Indeed, if $x \in \langle I, S\rangle$ then, by (β), $x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2)$ $(s_1, s_2 \in S, a_1, a_2 \in I)$ and choosing any u in $S \cap I$ we may suppose that $s_2 \leqq u \leqq s_1$, $a_2 \leqq u \leqq a_1$. Then,

$$x = (x \wedge s_1) \vee (x \wedge a_1) \equiv (x \wedge s_2) \vee (x \wedge a_1) = x \wedge a_1 (\theta[S]), \quad \text{for} \quad s_2 \leqq a_1,$$

and, similarly, $x = x \vee a_2 (\theta[S])$. For $y = (x \wedge a_1) \vee a_2$ we have $a_2 \leqq y \leqq a_1$ yielding $y \in I$ and $x \equiv x \vee a_2 \equiv (x \wedge a_1) \vee a_2 = y(\theta[S])$ proving the theorem.

THEOREM 3 (Second isomorphism theorem). *Let L be a lattice S a convex sublattice and T a standard sublattice of L such that $T \leqq S$. Then S is standard in L if and only if S/T is standard in L/T and in this case the isomorphism $L/S \cong (L/T)/(S/T)$ holds.*

Proof. If S is standard then S/T is standard in L/T by Lemma 3. The converse is proved in the same way as it is in [1] for standard ideals. The second isomorphism theorem for universal algebras finishes the proof.

It has been proved (see [1]) that L is a distributive lattice whenever every ideal of it is standard. A similar statement holds for standard sublattices.

THEOREM 4. *Let u be an element of the lattice L. If every convex sublattice containing u is standard then L is a distributive lattice.*

Proof. We shall prove that distributivity is implied whenever the ideals and the dual ideals containing u are standard. Let, namely, $L_1 = L/(u]$ and $L_2 = L/[u)$. The condition and Lemma 3 imply that each ideal of L_1 and each dual ideal of L_2 is standard. Thus, both L_1 and L_2 are distributive. If $a \leqq b$ have the same image both in L_1 and in L_2 then exist $p \leqq u \leqq q$ such that $b = a \vee p$ and $a = b \wedge q$, since both $(u]$ and $[u)$ are standard. Thus,

$$a = b \wedge q = b \wedge ((b \wedge q) \vee q) = b \wedge (a \vee q) \geqq b \wedge (a \vee p) = b \wedge b = b,$$

proving that L is a subdirect product of the two distributive lattices L_1 and L_2. Hence, L is itself distributive.

THEOREM 5. *In a relatively complemented lattice every congruence class is a standard sublattice.*

Proof. Let L be a relatively complemented lattice and let θ be a congruence relation on L. Let, further, S denote the congruence class containing a given element a of L. For $x \leqq y$, $x \equiv y(\theta)$ let x' denote the relative complement of x in the interval $[x \wedge a, y]$ and let y' denote the relative complement of y in the interval $[x, a \vee y]$. From $x \equiv y(\theta)$ follows $x \wedge a = x \wedge x' \equiv y \wedge x' = x'(\theta)$ yielding $a = a \vee (x \wedge a) \equiv a \vee x'(\theta)$. Hence, $t = a \vee x'$ is an element of S and so is, dually, the element $s = a \wedge y'$. Further, $x' \leqq y \wedge (a \vee x') = y \wedge t$ implies $y = x' \vee x \leqq (y \wedge t) \vee x \leqq y$ proving $y = (y \wedge t) \vee x$. We get $x = (x \vee s) \wedge y$ dually. Thus, condition (γ) of Theorem 1 is satisfied for S, i.e., S is standard.

Each standard sublattice is a class of a congruence relation. If the lattice is relatively-complemented this relation is unique and all classes are standard sublattices, i.e., $\theta[S] = \theta$ holds for each congruence class S of θ.

In the following example we will give a congruence θ of a lattice L such that $\theta[S] = \theta$ holds for each congruence class S of θ though none of these classes are standard.

Let $N^{(n)}$ denote a family of lattices isomorphic to N_5 for each integer n. The elements of $N^{(n)}$ will be denoted by o_n, a_n, b_n, c_n, i_n, respectively, where o is the smallest element i is the greatest element and $a < b$. There is an amalgam L of these lattices such that $o_n = c_{n-1}$ and $a_n = i_{n-1}$ and L contains no further elements. Now, the classes $S_n = \{o_n, a_n, b_n\}$ are classes of a congruence relation having the desired property.

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