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ON FINITELY GENERATED SIMPLE MODULAR LATTICES

by

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0. Introduction

R. WILLE asked the following question: does every finitely generated modular lattice contain a prime quotient? The answer is negative, as shown by the following

THEOREM. *There exists a finitely generated simple modular lattice of infinite length.*

The proof is based on the method of [1].

1. Preliminaries

Let Q be the chain of all rational numbers $\frac{k}{2^n}$, $0 \leq k \leq 2^n$, $n = 0, 1, 2, \ldots$.

LEMMA [1]. *Let N be a bounded distributive lattice. Then there exists a bounded modular lattice M with the following properties*:

(i) *M has three elements u_1, u_2, u_3 such that* 0, u_1, u_2, u_3, 1 *form a sublattice isomorphic to $\mathfrak{M}_5$, (the five element modular but not distributive lattice) and $(u_i]$ is isomorphic to N*;

(ii) *for every congruence relation Θ_i of $(u_i]$ $(i = 1, 2, 3)$ there exists exactly one congruence relation $\overline{\Theta}$ of M such that for b, c, $\in (u_i]$ $b \equiv c(\overline{\Theta})$ iff $b \equiv c(\Theta_i)$.*

We apply this lemma for the lattice $Q = N$ and M will always denote this lattice getting from Q.

The following lattice construction is due to HALL and DILWORTH [2] (see also [1]). Let us take two bounded lattices L_1 and L_2. Suppose that L_1 has a principal dual ideal I_1, L_2 has a principal ideal I_2 and $I_1 \cong I_2$ by $\varphi : x \to x'$. We get a lattice L as follows: L is the set of all $x \in L_1$, $y \in L_2$, we identify x with x' for all $x \in I_1$; $x \leq y$ have unchanged meaning if $x, y \in L_1$, or $x, y \in L_2$ and $x < y$, $x, y \notin I_1 = I_2$ iff $x \in L_1$, $y \in L_2$ and there exists a $z \in I_1$, such that $x < z$ in L_1 and $x < y$ in L_2. We denote L as follows:

$$L = L_1 + L_2(\varphi I_1 = I_2).$$

It is easy to see that for modular lattices L_1, L_2 the lattice L is modular, too ([1] Lemma 2).

2. The construction

In the lattice M the dual ideal $[u_i)$ is isomorphic to $(u_j]$ and both are isomorphic to Q. Let us take M in five disjoint replicas M_1, M_2, M_3, M_4 and M_5. We denote the elements $u_1, u_2, u_3, 0, 1$ in M_i by $u_1^i, u_2^i, u_3^i, 0^i, 1^i$. The $[u_1^1) \subseteq M_1$ is isomorphic to $(u_3^2] \subseteq M_2$, by the natural isomorphism. For a natural isomorphism we use always the symbol φ_n.

We can define the following lattices

$$A_0 = M_5 + M_4(\varphi_n[u_3^5) = (u_1^4]);$$

$$A_1 = M_1 + M_2(\varphi_n[u_1^1) = (u_3^2]);$$

$$A_2 = M_4 + A_1(\varphi[u_3^3) = (u_1^2])$$

where

$$\varphi^{-1}(x) = \begin{cases} \dfrac{x}{2} & \text{for } 0^1 \leq x \leq u_1^1, \\ \dfrac{x+1}{2} & \text{for } 0^2 \leq x \leq u_1^2, \end{cases}$$

$$K = A_0 + A_2(\varphi_n[1^4) = (0^3]).$$

The poset of all $0^i, u_1^i, u_i^2, u_3^i, 1^i \in K$ $(i = 1, 2, 3, 4, 5)$ is represented by Fig. 1.

In the lattice M, $u_i/0$ and $1/u_j$ are projective quotients, hence $1^3/u_3^3$ and $u_1^2/0^2$ are projective in K. We shall denote the corresponding algebraic function which maps $1^3/u_3^3$ onto $u_1^2/0^2$ by $f_0(x)$. Then

$$f_0(x) = \{\{[[((((x \wedge u_2^3) \vee u_1^3) \wedge u_3^5) \vee u_2^5) \wedge u_1^5] \vee u_2^4] \wedge u_3^4\} \vee u_2^2\} \wedge u_1^2.$$

Similarly, $1^3/u_3^3$ and $u_1^1/0^1$ are projective, and there corresponds to the algebraic function $g_0(x)$. $f_0^{-1}(x)$ and $g^{-1}(x)$ are inverse functions.

$u_1/0^2$ is a sublattice of $1^3/u_3$, therefore the restrictions of $f_0, f_0^{-1}, g_0, g_0^{-1}$ define four unary partial operations $\tilde{f}_0, \tilde{f}_0^{-1}, \tilde{g}_0, \tilde{g}_0^{-1}$ on $1^3/u^3$. On the other hand, we have a natural isomorphism between $1^3/u_3^3$ and Q. By this isomorphism we get the partial operations f, f^{-1}, g, g^{-1} on Q corresponding $\tilde{f}_0, \tilde{f}_0^{-1}, \tilde{g}_0, \tilde{g}_0^{-1}$. By the definition of A_2 we have:

$$f(x) = \frac{x+1}{2},\ x \in Q;\quad f^{-1}(x) = 2x - 1,\quad \frac{1}{2} \leq x \leq 1;$$

$$g(x) = \frac{x}{2},\ x \in Q;\quad g^{-1}(x) = 2x,\quad 0 \leq x \leq \frac{1}{2}.$$

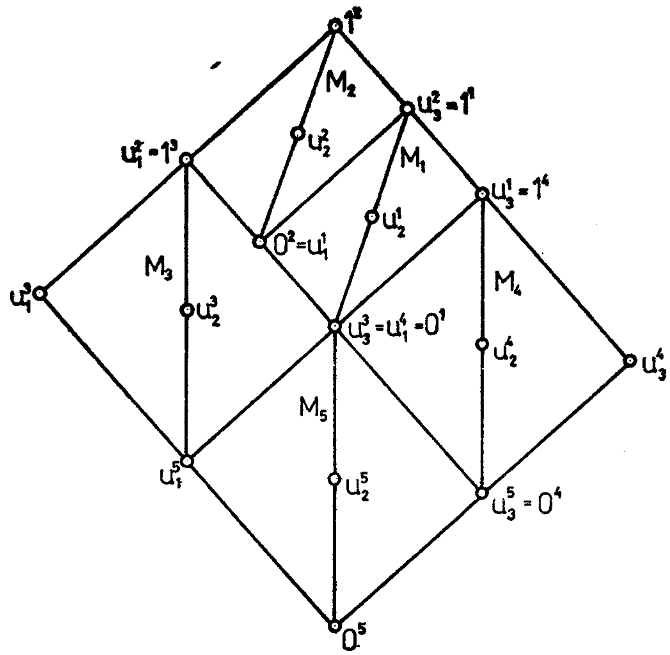

Fig. 1

We have to prove that K is a finitely generated simple lattice. First we prove that $P = \{u_1^i, u_2^i, u_3^i \mid i = 1, 2, \ldots, 5\}$ is a generating set. We can see that $f_0, f_0^{-1}, g_0, g_0^{-1}$ are defined by the elements of P, hence it is enough to prove that in the partial algebra

$$Q = < Q;\ \vee,\ \wedge,\ f, f^{-1}, g,\ g^{-1} >$$

the subset $Q_0 = \{0, 1\}$ is a generating set. Let $\bar{Q}_0$ be denote the subalgebra generated by Q_0, and let Q_n be the following subset of Q:

$$Q_n = \left\{\frac{0}{2^n}, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{k}{2^n}, \ldots, \frac{2^n}{2^n}\right\}.$$

The union $\bigcup_{n=0}^{\infty} Q_n$ is obviously Q. We prove by induction for n, that every $Q_n \subseteq \bar{Q}_0$; $Q_0 \subseteq \bar{Q}_0$ by the definition of $\bar{Q}_0$. Now let $Q_n \subseteq \bar{Q}_0$ and let $\frac{u}{2^{n+1}} \in Q_{n+1}$, where $0 \leqq u \leqq 2^{n+1}$. If $u \leqq 2^n$ then $\frac{u}{2^n} \in Q_n$ and $\frac{u}{2^{n+1}} = g\left(\frac{u}{2^n}\right)$, hence $\frac{u}{2^{n+1}} \in$ $\in Q_{n+1}$. In other case $2^n \leqq u \leqq 2^{n+1}$ we take $\frac{u - 2^n}{2^n} \in Q_n$ and apply the operation f getting $\frac{u}{2^{n+1}} = f\left(\frac{u-2^n}{2^n}\right) \in Q_{n+1}$.

By condition (ii) of the Lemma every congruence relation of K is the smallest extension of a congruence relation of the quotient $1^3/u_3^1$. Therefore if Q is a simple partial algebra, K is a simple lattice. Let u, v, $u < v$ be two elements of Q and let Θ be a congruence relation such that $u \equiv v(\Theta)$. Then there exist two integers k and n with the property

$$u \leq \frac{k}{2^n} < \frac{k+1}{2^n} \leq v.$$

From $u \equiv v(\Theta)$ we get $\frac{k}{2^n} \equiv \frac{k+1}{2^n}(\Theta)$. If $\frac{k+1}{2^n} \leq \frac{1}{2}$ then we can apply g^{-1}, hence

$$\frac{k}{2^{n-1}} = g^{-1}\left(\frac{k}{2^n}\right) \equiv g^{-1}\left(\frac{k+1}{2^n}\right) = \frac{k+1}{2^{n-1}}(\Theta).$$

In the other case $\frac{1}{2} \leq \frac{k}{2^n}$; from $\frac{k}{2^n} \equiv \frac{k+1}{2^n}(\Theta)$ we get using f^{-1}

$$\frac{k-2^{n-1}}{2^{n-1}} = f^{-1}\left(\frac{k}{2^n}\right) \equiv f^{-1}\left(\frac{k+1}{2^n}\right) = \frac{k+1-2^{n-1}}{2^{n-1}}(\Theta).$$

By induction we have that from $u \equiv v(\Theta)$ it follows $0 \equiv 1(\Theta)$, i.e., Q is a simple partial algebra.

This completes the proof of the theorem.

COROLLARY. *There exist a finitely generated modular lattice which does not contain a prime quotient.*

PROOF. Let K be a finitely generated simple modular lattice of infinite length. If a/b is a prime quotient of K and c/d is an arbitrary quotient $c \equiv d(\Theta(a, b))$, hence there exists a finite chain $d = z_0 < z_1 < \ldots < z_n = c$ such that z_i/z_{i-1} is weak projective into a/b $(i = 1, 2, \ldots, n)$ and therefore z_i/z_{i-1} has a finite length. This implies that c/d has a finite length. This is a contradiction to the assumption that K is of infinite length.

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[1] E. T. SCHMIDT, Every finite distributive lattice is the congruence lattice of some modular lattice, *Algebra Universalis* **4** (1974), 49–57.
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