A REMARK ON LATTICE VARIETIES DEFINED
BY PARTIAL LATTICES

by

E. T. SCHMIDT

A lattice variety is a class of lattices, which is closed under the formation of direct products, subalgebras and homomorphic images.

It is well known that the two most important lattice varieties, the variety of modular lattices $\mathscr{M}$, and the variety of distributive lattices $\mathscr{D}$ have every useful characterizations with sublattices. These lattices N_5 and M_5 are given in Figure 1.
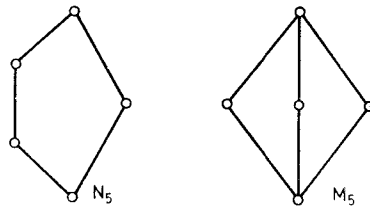

Fig. 1

A lattice L is distributive iff L has no sublattice isomorphic to M_5 or N_5; a lattice L is modular iff it has no sublattice isomorphic to N_5; a modular lattice is distributive iff it has no sublattice isomorphic to M_5.

A similar characterization of lattice varieties is due from R. WILLE [4]. He defined the notation of primitive subset: a finite subset P of a lattice L is called *primitive* if $\bigwedge_{\substack{c,d \in P \\ c \nleq d}} \Theta(c \vee d, d) > \omega$, i.e. if there exists a proper quotient a/b of L such that a/b is weak projective into $c \vee d/d$ for all $c, d \in P$, $c \nleq d$.

WILLE has proved the following

THEOREM (WILLE [4]). *Let $\mathscr{P}$ be a set of finite posets, V a variety of lattices and $V_{\mathscr{P}}$ the class of all lattices in V having no primitive subset o-isomorphic[1] to a member of $\mathscr{P}$. Then $V_{\mathscr{P}}$ is a variety.*

N_5 and M_5 are subdirectly irreducible lattices, therefore if these lattices are sublattices of a lattice L then N_5, M_5 form primitive subsets of L. We have therefore: if L is a nonmodular lattice, then L contains a primitive subset which is o-isomorphic to N_5. The converse statement is not true, the lattice K (Figure 2) is a (simple) modular lattice and the elements $0, a, b, c, 1$ form a primitive subset o-isomorphic to N_5.

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[1] Order-isomorphic.

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It is easy to see that a modular lattice L is distributive iff L does not contain a primitive subset o-isomorphic to M_5. (The primitive subsets of a distributive lattice are namely the two element subchains.) For the variety $\mathscr{D}$ we have also two possibilities, we can take M_5 as a sublattice or as a primitive subset.
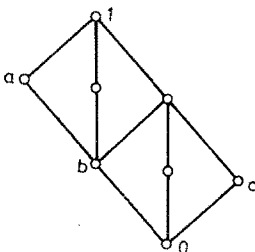

Fig. 2

We know a third characterization of lattice varieties, the characterization of n-distributive lattices.

A modular lattice satisfying the identity

$$x \vee \bigwedge_{i=0}^{n} y_i = \bigwedge_{j=0}^{n} \left(x \vee \bigwedge_{\substack{i=0 \\ i \neq j}}^{n} y_i \right)$$

is called a n-distributive lattice (A. HUHN [3]). Huhn has proved that the modular lattice L is n-distributive iff L does not contain a sublattice $B \cong 2^{n+1}$ and element x such that x is the relative complement of all atoms of B in interval [inf B, sup B].

This characterization is different from the first two characterizations, the element x and the sublattice B can generate different sublattices, therefore this is not a "sublattice characterization". On the other hand the poset $B^* = \{x, B\}$ is primitive, but it is not possible to take B^* as a poset; the following counter-example is from A. HUHN (unpublished). Let L be the lattice represented by Fig. 3.

L is a 2-distributive lattice; the elements u, a, b, c, d, e, f, 1 form a poset which is o-isomorphic to 2^3, and x is a relative complement of a, b, c in interval $[u, 1]$.

All these characterizations are similar. In the first case we have (primitive) *sublattices,* in the second case (primitive) *subsets* and in the last case (primitive) *partial lattices.*

In this note we shall give a characterization of lattice varieties with partial lattices.

Let L be a lattice, and $\emptyset \neq P \subseteq L$. Suppose we have two partial binary operations $\vee'$, $\wedge'$ defined P such that if $a \vee' b = u$, $c \wedge' d = v$, then $a \vee b = u$, $c \wedge d = v$ in L. We say that $\mathscr{P} = \langle P, \vee', \wedge' \rangle$ is a *weak sublattice* of L. (See [1] p. 81.) This $\mathscr{P}$ is *weak partial lattice* (for the definition we refer to [2]), and every weak partial lattice is a weak sublattice of a lattice L.

Let $\mathscr{P}$ be a weak partial lattice and V a lattice variety. Take the class $V_{\mathscr{P}}$ of all lattices L in V having no weak sublattice $\mathscr{P}'$ with the following two properties:

(i) $\mathscr{P}'$ is isomorphic to $\mathscr{P}$;

(ii) P' is a primitive subset of L.

$V_{\mathscr{P}}$ is obviously closed under the formation of subalgebras. We prove that $V_{\mathscr{P}}$ is closed under the formation of direct product. Let $L=\Pi\ (L_i;\ i\in I)$ the direct product of the lattices $L_i\in V_{\mathscr{P}}$. If L contains a weak sublattice $\mathscr{P}'$ isomorphic to $\mathscr{P}$, and P' is a primitive subset of L, then there exists a proper quotient a/b of L such that a/b is weak projective into $c\vee d/d$ for all $c,d\in P',\ c\not\equiv d$. Let φ_i be the projec-
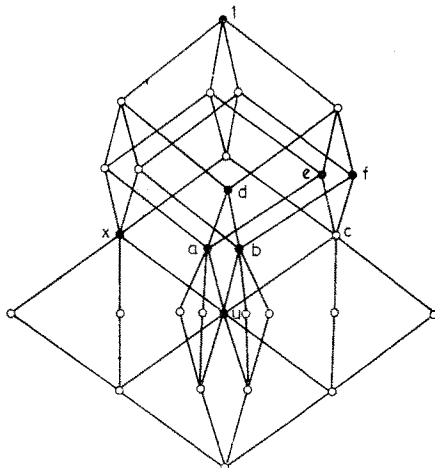

Fig. 3

tion from L onto L_i. We denote by Θ_i the congruence relation of L corresponding to φ_i. $L=\Pi\ (L_i;\ i\in I)$ implies $\wedge(\Theta_i;\ i\in I)=\omega$, hence there is an $i\in I$ such that $\varphi_i(a)\neq$ $\neq\varphi_i(b)$. The set $\varphi_i(P)=\{\varphi_i(p)|p\in P\}\subseteq L_i$ is therefore a weak subalgebra of L_i, which is isomorphic to P. By Lemma 5 of [4], $\varphi_i(P)$ is a primitive subset and so $\varphi_i(P)$ has the properties (i)—(ii). Thus $L_i\notin V_{\mathscr{P}}$, which is a contradiction.

$V_{\mathscr{P}}$ is not always closed under formation of homomorphic images. (It is easy to give two lattices L, L' such that L' is a homomorphic image of L, L' is subdirectly irreducible, and L does not contain a sublattice isomorphic to L'.)

We need the following

Definition. The partial lattice $\mathscr{P}'$ has concerning V the property (H) if for any homomorphism

$$\varphi: L \to L' \quad (L, L'\in V)$$

if $\mathscr{P}'$ is a weak sublattice of L' then there exists a weak sublattice $\mathscr{P}$ of L isomorphic to $\mathscr{P}'$, and φ maps $\mathscr{P}$ onto $\mathscr{P}'$.

We prove

THEOREM. *Let $\mathscr{P}$ be a finite weak partial lattice and V a lattice variety. If $\mathscr{P}$ has the property (H) concerning V, then the class $V_{\mathscr{P}}$ of all lattices L in V having no weak sublattice $\mathscr{P}'$ satisfying* (i) *and* (ii) *form a lattice variety.*

PROOF. We must prove that $V_{\mathscr{P}}$ is closed under formation of homomorphism. Let $L\to L'$ be a homomorphism and $L, L'\in V$. Let $\mathscr{P}'$ be a weak sublattice of L'