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On the length of the congruence lattice of a lattice

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In [1] J. Berman has shown that for every chain L of length n there exists a finite lattice K such that $L \cong \Theta(K)$ and K has length 5. With a similar construction we prove:

THEOREM. *Let L be a finite distributive lattice with exactly one dual atom. Then there exists a finite lattice K such that $L \cong \Theta(K)$ and K has length* 5.

First we prove

LEMMA. *Let k be an arbitrary natural number. There exists a lattice T^k of length four such that*

(i) *for every i $(1 \leqslant i \leqslant k)$ T^k has three elements a^i, b^i, c^i; $0 \prec a^i \prec b^i \prec c^i$, $c^i \wedge c^j = 0$ $(i \neq j)$;*

(ii) *T^k has exactly one non-trivial congruence relation Θ for which $\Theta = \Theta(a^i, b^i)$ $(i = 0, 1, \ldots, k)$ and the only non-trivial Θ-classes are $\{a^i, b^i\}$ $(i = 0, 1, \ldots, k)$;*

(iii) *Every a^i, b^i, c^i $(i = 1, 2, \ldots, k)$ is join-irreducible.*

Proof. Take the following lattice represented by Fig. 1.

It is easy to see that $\Theta(0, x) = \Theta(y, 1) = \iota$ for every $x > 0$, $y < 1$, and $\Theta(b^i, c^i) = \Theta(d^i, 1)$, $\Theta(b^i, d^i) = \Theta(c^i, 1)$, $\Theta(b^i, b^0) = \Theta(c^i, 1)$. The intervals $[a^i, b^i]$ and $[a^j, b^j]$ $(i \neq j)$ are projective, hence the equivalence relation Θ defined by the following classes $\{a^i, b^i\}$ $(i = 0, 1, \ldots, k)$ and $\{x\}$ for all $x \in T^k$, $x \neq a^i$, $x \neq b^i$ $(i = 0, 1, \ldots, k)$ is the only one non-trivial congruence relation of T^k. (i) and (iii) are obviously satisfied.

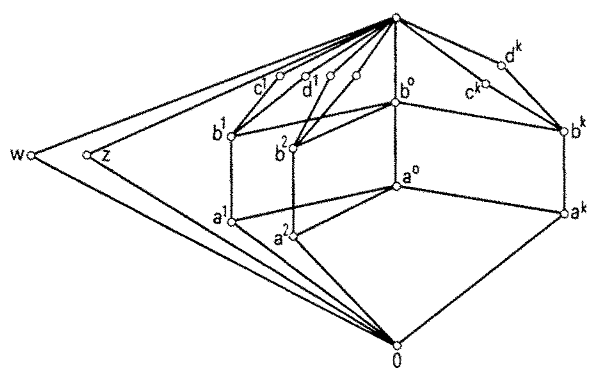

Fig. 1.

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The proof of the theorem. Let L be a finite distributive lattice with exactly one dual atom and $J(L)=\{p_1, \ldots, p_n\}$ denote the poset of all non-unit, non-zero join-irreducible elements of L. Then L is completely determined by $J(L)$. We can assume $J(L)\neq\emptyset$, for $J(L)=\emptyset$ we get that L is the 2-element lattice and then an arbitrary simple lattice K of length 5 has the property $L\cong\Theta(K)$.

Take the lattice T^n in n copies $T_1^n, T_2^n, \ldots, T_n^n$ where $T_i^n=\{0, 1, w_i, z_i, a_i^0, b_i^0\}\cup\cup\bigcup_{j=1}^n\{a_i^j, b_i^j, c_i^j, d_i^j\}$. We identify the unit elements and the zero elements of these lattices and define two elements x, y such that $x\wedge y=0$, $x\vee y=1$, $x\wedge a=y\wedge a=0$,

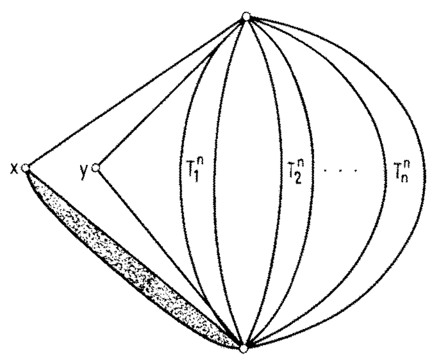

Fig. 2.

$x\vee a=y\vee a=1$ for every $a\in\bigcup_{i=1}^n T_i^n$, $a\neq 0, 1$. We define some further elements under x such that the ideal $(x]$ will be a simple lattice of length four. In this way we get the poset $K=(x]\cup\{y\}\cup\bigcup_{i=1}^n T_i^n$, which is obviously a lattice (Fig. 2).

Let Ω be the set of all pairs (i, j) $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n)$ such that $p_i \succ\!\!- p_j$ in $J(L)$. For each $(i, j)\in\Omega$ we adjoin two new elements u_{ij} and v_{ij} to K with the following covering diagram:

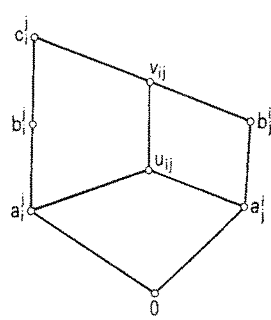

Fig. 3.

In this way we get from K the poset $K^*=K\cup\bigcup_{(i,j)\in\Omega}\{u_{ij}, v_{ij}\}$.

We define

$$A_{ij}=\begin{cases}\{0,\ a_i^j,\ b_i^j,\ c_i^j,\ a_j^i,\ b_j^i,\ u_{ij},\ v_{ij}\} & \text{if} \quad (i,j)\in\Omega,\\ \{0,\ a_i^j,\ b_i^j,\ c_i^j\} & \text{if} \quad (i,j)\notin\Omega.\end{cases}$$

It is easy to verify that K^* is a lattice and that the A_{ij} are ideals of K^*. Also $A_{ij}\cap A_{kl}=\{0\}$ for $(i,j)\neq(k,l)$.

We shall describe the congruence relations of K^*. The join-irreducible congruence relations of K^* are $\Theta(s,t)$ where $s\prec t$ in K^*. By the lemma we can see that if $\Theta(s,t)\neq\iota$ then for s and t the only possible choices are $\{a_i^j, b_i^j\}$ $(0\leqslant j\leqslant n,\ 1\leqslant i\leqslant n)$, $\{u_{ij}, v_{ij}\}$, or $\{v_{ij}, c_i^j\}$ if $(i,j)\in\Omega$. Using the lemma again we have $\Theta(a_i^j, b_i^j)=\Theta(a_i^0, b_i^0)$. $\Theta(v_{ji}, u_{ji})=\Theta(a_j^0, b_j^0)$ and $\Theta(v_{ij}, c_i^j)=\Theta(a_i^0, b_i^0)$ and hence there are most n non-trivial join-irreducible congruences in $\Theta(K^*)$: $\Theta(a_1^0, b_1^0),\ldots,\Theta(a_n^0, b_n^0)$. If $(i,j)\in\Omega$ then from $a_i^j\equiv b_i^j$ $(\Theta(a_i^0, b_i^0))$ we get $u_{ij}=u_{ij}\vee a_i^j\equiv u_{ij}\vee b_i^j=c_i^j$ $(\Theta(a_i^0, b_i^0))$, hence $a_j^i=u_{ij}\wedge b_j^i\equiv c_i^j\wedge b_j^i=b_j^i$ $(\Theta(a_i^0, b_i^0))$. Thus we get that $(i,j)\in\Omega$ implies $\Theta(a_i^0, b_i^0)\geqslant\Theta(a_j^0, b_j^0)$.

We shall characterize the congruence relations $\Theta(a_i^0, b_i^0)$. By inspection $\Theta(a_i^0, b_i^0)$-classes are:

$$\{a_j^t, b_j^t\}, \quad t=0,1,\ldots,n, \tag{1}$$

$$\{u_{sj}, v_{sj}\} \quad \text{if} \quad (s,j)\in\Omega, \tag{2}$$

$$\{u_{jl}, v_{jl}, c_j^l\} \quad \text{for} \quad (j,l)\in\Omega, \tag{3}$$

where $i=j$ or there exists a sequence $m_1, m_2,\ldots, m_r$, such that (i,m_1), $(m_1,m_2),\ldots$, $(m_r,j)\in\Omega$. It follows that $\Theta(a_1^0, b_1^0)$, $\Theta(a_2^0, b_2^0),\ldots,\Theta(a_n^0, b_n^0)$ are different congruence relations. The correspondence $p_i\to\Theta(a_i^0, b_i^0)$ is therefore a poset isomorphism from $J(L)$ to $J(\Theta(K^*))$; thus $L\cong\Theta(K^*)$, since $1\in L$ and $\iota\in\Theta(K^*)$ are both join-irreducible. The length of K^* is 5.

PROBLEM. Does there exist to every finite distributive lattice L with n dual atoms a natural number $\varphi(n)$ such that $L\cong\Theta(K)$ for some finite lattice K of length $\varphi(n)$? (Conjecture $\varphi(n)=5n$.)

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REFERENCE

[1] J. Berman, *On the length of the congruence lattice of a lattice*, Alg. Univ. *2* (1972), 18–19.

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