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EVERY FINITE DISTRIBUTIVE LATTICE IS THE CONGRUENCE LATTICE OF SOME MODULAR LATTICE

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1. Introduction

The purpose of this paper is to prove the theorem formulated in the title. The notation used is that of Grätzer [2]. The unary algebraic functions play by the description of congruence relations a very important role. Let $p=p(x)$ be an unary algebraic function on the modular lattice K, and let $a_0 \leqslant b_0$ be two elements of K; it is easy to show that there exists a pair $a, b \in K$, $a_0 \leqslant a \leqslant b \leqslant b_0$ such that the restriction of p to $[a, b]$, $p|_{[a,b]}$ is an isomorphism between $[a, b]$ and $[p(a_0), p(b_0)]$, i.e., these intervals are projective. Let now f: $[a, b] \to [c, d]$ be an arbitrary isomorphism, then we take f as a partial unary operation with the domain $[a, b]$. We called such a partial operation a *-operation. The inverse f^{-1} of f is again a *-operation. If there exists for f an unary algebraic function p on K, such that $f=p|_{[a,b]}$, then p is called a *realization* of f.

Now we consider a sublattice K_0 of the lattice K (in other words K is an extension of K_0). Then K determines a system of projective intervals $[a_\alpha, b_\alpha]$, $[c_\alpha, d_\alpha]$ $(\alpha \in \Omega)$ of K_0. Let p_α denote the unary algebraic function which maps $[a_\alpha, b_\alpha]$ into $[c_\alpha, d_\alpha]$. With the corresponding *-operation $\tilde{p}_\alpha = p_\alpha|_{[a_\alpha, b_\alpha]}$ we get a partial algebra $K_0^* = \langle K_0;\ \vee,\ \wedge,\ \tilde{p}_\alpha \mid \alpha \in \Omega \rangle$. A congruence relation of K_0 has an extension to K iff Θ is a congruence relation of K_0^*.

DEFINITION. Let K_0 be a lattice and $f_\alpha : [a_\alpha, b_\alpha] \to [c_\alpha, d_\alpha]$ be *-operations of K_0. If there exist an extension K of K_0 such that the following conditions are satisfied:

(1) every f_α has a realization in K;

(2) for every $\Theta \in \mathbf{C}(K_0^*)$ there exists exactly one congruence relation $\bar{\Theta}$ of K such that for $a, b \in K_0$, $a \equiv b(\bar{\Theta})$ iff $a \equiv b(\Theta)$, then we say that K is a *realization* of K_0^*.

For a realization K of K_0^*, $\mathbf{C}(K) \cong \mathbf{C}(K_0^*)$ obviously holds, i.e., the congruence lattices are isomorphic.

EXAMPLE. Let K_0 be the three element chain: $0 < a < 1$. Then $[0, a]$ and $[a, 1]$ are isomorphic, so we have *-operations f: $[a, 1] \to [0, a]$ and f^{-1}. A realization of $\langle K_0;\ \vee,\ \wedge, f, f^{-1} \rangle$ is the following lattice, where $p(x) = \{[(x \vee b) \wedge c] \vee d\} \wedge a$ realizes f:

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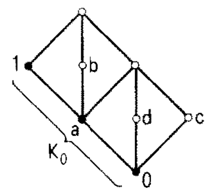

Fig. 1

We ask: does there exist for a modular lattice K_0 and *-operations a realization which is modular too? We prove, if K_0 is a chain then there always exists such a realization. In the second paragraph we give for a finite distributive lattice L a chain K_0 and *-operations f_i $(i=1, 2,\ldots)$ with the property $L\cong \mathbf{C}(K_0^*)$, where $K_0^*= =\langle K_0; \vee, \wedge, f_i\rangle$. In the third paragraph there are proved two lemmas, and in the last paragraph the construction of K is given, which realizes K_0^*.

2. A partial algebra

Let Q be the chain of all rational numbers r with $0\leqslant r\leqslant 1$. Two non-trivial intervals $[a, b]$ and $[c, d]$ of Q are isomorphic, hence an arbitrary isomorphism defines a *-operation $f: [a, b]\to [c, d]$.

A congruence relation Θ is called irreducible if it is a join-irreducible element of the congruence lattice. The smallest congruence relation Θ such that $a\equiv b(\Theta)$ will be denoted by $\Theta(a, b)$ and is called a principal congruence relation. If the congruence lattice is finite then every irreducible congruence relation is obviously principal.

THEOREM 1. *Let L be a finite distributive lattice. Then we can define on Q *-operations $f_1, f_2, \ldots$ such that the congruence lattice of the partial algebra $Q_L= \langle Q; \vee, \wedge, f_i \mid i=1, 2, \ldots\rangle$ will be isomorphic to L.*

Proof. We prove the theorem by induction as follows: for each positive integer n let $P(n)$ be the assertion that every distributive lattice of length $\leqslant n$ is isomorphic to $\mathbf{C}(Q_L)$ for some partial algebra Q_L defined on Q with *-operations. If we take on Q for each pair $0\leqslant a_i<b_i\leqslant 1$ an arbitrary isomorphism $f_i: [a_i, b_i]\to [0, 1]$, then the corresponding partial algebra $Q_2=\langle Q; \vee, \wedge, f_i\rangle$ is obviously simple, i.e. $\mathbf{C}(Q_2)\cong 2$. $P(1)$ is proved.

We shall show that $P(n-1)$ implies $P(n)$. Now let L be any distributive lattice of length n (>1) and let p denote a maximal irreducible element of L. Let $p_1, p_2, \ldots, p_k$ denote those irreducible elements of L which are covered by p in the poset of the irreducible elements. If d denotes the join of all irreducible elements of L different from p, then the length of the ideal $L_1=(d]$ is $n-1$. By the induction hypotheses there exists a partial algebra Q_{L_1} defined on $Q=[0, 1]$ with *-operations f_i, such that $\mathbf{C}(Q_{L_1})\cong L_1$. Now we take the interval $[0, 2]$ of rational numbers and we define the partial algebra Q_L on the set $[0, 2]$.

The congruence relations of Q_{L_1} which correspond to $p_i \in L_1$ $(i=1,\ldots,k)$ are irreducible, consequently principal, i.e. $p_i \rightarrow \Theta(a_i, b_i)$ $(a_i<b_i)$. We distinguish two cases. First, if p is an atom $(k=0)$, then we take for every pair $1 \leqslant a_i < b_i \leqslant 2$ an arbitrary isomorphism $f_i': [a_i, b_i] \rightarrow [1, 2]$, and let

$$Q_L=\langle[0, 2];\ \vee,\ \wedge, f_i, f_i' \mid i=1, 2,\ldots\rangle.$$

Then every congruence relation of Q_{L_1} is remade a congruence of Q_L if we take the rational numbers $1<r\leqslant 2$ as one element classes. For every $1\leqslant a_i<b_i\leqslant 2$ there is $\Theta(a_i, b_i)=\Theta(1, 2)$ and therefore $\Theta(1, 2)$ is an atom. Thus we have $\mathbf{C}(Q_L)\cong\mathbf{C}(Q_{L_1})\times\times 2\cong L$. The intervals $[0, 1]$ and $[0, 2]$ are isomorphic, we can take also Q_L as a partial algebra defined on $[0, 1]$.

The second case is when $k>0$. We take the following isomorphism:

$$g_0:[1, 2]\rightarrow\left[1+\frac{k}{k+1}, 2\right] \tag{1}$$

is an arbitrary isomorphism with the property that $\lim_{t\rightarrow\infty} g_0^t(1)=2$. (For instance the mapping $x\rightarrow(x+2k)/(k+1)$ is such an isomorphism);

$$g_i:[a_i, b_i]\rightarrow\left[1+\frac{i-1}{k+1}, 1+\frac{i}{k+1}\right], \quad i=1, 2,\ldots, k \tag{2}$$

is an arbitrary isomorphism. The partial algebra Q_L is defined by

$$Q_L=\langle[0, 2];\ \vee,\ \wedge, f_j, g_i, g_i^{-1} \mid j=1, 2,\ldots;\ i=0, 1,\ldots, k\rangle.$$

We shall prove that $\mathbf{C}(Q_L)$ is isomorphic to L. To do this we prove some simple statements:

1. every $\Theta\in\mathbf{C}(Q_{L_1})$ has an extension to a congruence relation $\bar{\Theta}$ of Q_L.

Proof. Let Θ be an arbitrary congruence relation of Q_{L_1}, Θ defines a reflexive and symmetric relation Θ^* on $[0, 2]$:

$$u\equiv v(\Theta^*) \quad \text{iff} \quad \begin{cases} \text{either } 0\leqslant u, v\leqslant 1 \text{ and } u\equiv v(\Theta) \quad \text{or} \\ u=g_0^s g_i(x),\ v=g_0^s g_i(y),\ g_i\leqslant x, y\leqslant b_i,\ x\equiv y(\Theta) \\ \text{for some integer } s\geqslant 0 \text{ and } 1\leqslant i\leqslant k, \text{ where } g_0^0 \text{ is the} \\ \text{identity map.} \end{cases}$$

Let $\bar{\Theta}$ denote the transitive extensions of Θ^*. The restriction of $\bar{\Theta}$ to $[0, 1]$ is Θ, and $\bar{\Theta}$ is obviously a congruence relation of Q_L. $\bar{\Theta}$ is therefore an extension (the smallest extension) of Θ to Q_L.

2. Every congruence relation $\Theta(u, v)$, $0 \leqslant u \leqslant v < 2$ of Q_L is the extension of a congruence relation $\Theta \in \mathbf{C}(Q_{L_1})$.

Proof. Let x, y be two elements of the interval $[a_i, b_i]$ and let ϕ be an arbitrary congruence relation of Q_L. For two integers s and i $(1 \leqslant i \leqslant k)$. $x' = g_0^s g_i(x) \equiv \equiv g_0^s g_i(y) = y'(\phi)$ if and only if $x = g_i^{-1} g_0^{-s}(x') \equiv g_i^{-1} g_0^{-s}(y') = y(\phi)$. $\Theta(x', y') \in \mathbf{C}(Q_L)$ is therefore an extension of $\Theta(x, y) \in \mathbf{C}(Q_{L_1})$. If $1 \leqslant u < v < 2$, then there exists a natural number m such that $u, v \leqslant g_0^m(1)$. There exists a finite chain

$$u = u_0 < u_1 < \cdots < u_t = v$$

such that for every j $(=1, \ldots, t)$ $u_{j-1}, u_j \in [g_0^s g_i(a_i), g_0^s g_i(b_i)]$ for some s and i $(1 \leqslant i \leqslant k)$. We have proved that $\Theta(u_{j-1}, u_j)$ $(j = 1, \ldots, t)$ is the extension of a congruence relation relation $\Theta_j \in \mathbf{C}(Q_{L_1})$. Then $\Theta(u, v)$ is obviously the extension of $\bigvee_{j=1}^t \Theta_j$.

3. $\Theta(u, 2) = \Theta(1, 2)$ for every $1 \leqslant u < 2$, hence $\Theta(1, 2)$ is irreducible.

Proof. Let t be the last integer with $u \leqslant g_0^t(1)$. If $2 \equiv u(\phi)$ then $2 = g_0^{-t}(2) \equiv \equiv g_0^{-t}(u) = 1(\phi)$, hence $\Theta(u, 2) \geqslant \Theta(1, 2)$. But $\Theta(u, 2) \leqslant \Theta(1, 2)$ is trivially satisfied and thus $\Theta(u, 2) = \Theta(1, 2)$. If $\Theta(1, 2) = \Theta_1 \vee \Theta_2$ then there exists a sequence $2 = u_1 > u_2 > \cdots > u_r = 1$ such that for each i $u_i \equiv u_{i+1}(\Theta_1)$ or $u_i \equiv u_{i+1}(\Theta_2)$. For instance $2 \equiv u_2(\Theta_1)$. But $\Theta(1, 2) = \Theta(u_2, 2)$ implies $\Theta_1 = \Theta(1, 2)$; $\Theta(1, 2)$ is therefore irreducible.

4. For an irreducible congruence relation $\Theta \in \mathbf{C}(Q_{L_1})$ is $\Theta(1, 2) \geqslant \bar{\Theta}$ if and only if $\Theta \leqslant \Theta(a_i, b_i)$ for some $i \in \{1, \ldots, k\}$.

Proof. From $1 + (i-1)/(k+1) \equiv 1 + i/(k+1)$ $(\Theta(1, 2))$ we get by the applications of g^{-1} $a_i = g^{-1}(1 + (i-1)/(k+1)) \equiv g^{-1}(1 + i/(k+1)) = b_i(\Theta(1, 2))$ i.e. $\overline{\Theta(a_i, b_i)} \leqslant \Theta(1, 2)$. The statement 'only if' is trival.

1–4 imply that the poset of all irreducible congruence relations of $\mathbf{C}(Q_L)$ is isomorphic to the poset of all irreducible elements of L, following $\mathbf{C}(Q_L) \cong L$.

3. Two preliminary constructions

LEMMA 1. *Let N be a bounded distributive lattice. Then there exists a bounded modular lattice M with the following properties:*

(i) *M has three elements $\alpha_1, \alpha_2, \alpha_3$ such that $0, \alpha_1, \alpha_2, \alpha_3, 1$ form a sublattice isomorphic to $\mathcal{M}_5$ and $(\alpha_i]$ is isomorphic to N;*

(ii) *for every congruence relation Θ_i of $(\alpha_i]$ $(i = 1, 2, 3)$ there exists exactly one congruence relation $\bar{\Theta}$ of M such that for $\beta, \gamma \in (\alpha_i]$ $\beta \equiv \gamma(\bar{\Theta})$ iff $\beta \equiv \gamma(\Theta_i)$.*

Proof. We take the set M of all triples (x, y, z) $(x, y, z \in N)$ with the property $x \wedge y = x \wedge z = y \wedge z$. $(x, y, z) \leqslant (x', y', z')$ means that $x \leqslant x'$, $y \leqslant y'$ and $z \leqslant z'$; then M will be a poset. If $\beta = (x, y, z)$, $\gamma = (x', y', z') \in M$ then

$$(x \wedge x') \wedge (y \wedge y') = (x \wedge x') \wedge (z \wedge z') = (y \wedge y') \wedge (z \wedge z'),$$

therefore $\beta \wedge \gamma = (x \wedge x', y \wedge y', z \wedge z') \in M$. M is therefore a $\wedge$-semilattice. It is easy to prove – using the distributivity of N – that $\sup\{\beta, \gamma\} = \beta \vee \gamma$ exists and

$$\beta \vee \gamma = ((x \vee x') \vee [(y \vee y') \wedge (z \vee z')],$$
$$[(y \vee y') \vee [(x \vee x') \wedge (z \vee z')], (z \vee z') \vee [(x \vee x') \wedge (y \vee y')]).$$

The operations $\wedge$ and $\vee$ make M into a lattice. This lattice was defined first in [5].

Now we prove the modularity of M. Let be $\gamma = (x, y, z)$, $\beta = (u, v, w)$ and $\alpha = (a, b, c)$, $\alpha > \beta$. The modularity means $\alpha \wedge (\beta \vee \gamma) \leqslant \beta \vee (\alpha \wedge \gamma)$. Take the first components of these elements: $a \wedge \{(u \vee x) \vee [(v \vee y) \wedge (w \vee z)]\}$ and $(u \vee (a \wedge x)) \vee \{[v \vee (b \wedge y)] \wedge [w \vee (c \wedge z)]\}$. Then using the facts: $u \wedge w = u \wedge v$, $y \wedge z = x \wedge z$, $b \wedge v = v$, $c \wedge w = w$ and the distributivity of N we can write:

$$\begin{aligned}
a \wedge \{(u \vee x) \vee [(v \vee y) \wedge (w \vee z)]\} &= [a \wedge (u \vee x)] \vee [a \wedge (v \vee y) \wedge (w \vee z)] \\
&= [u \vee (a \wedge x)] \vee (a \wedge v \wedge w) \vee (a \wedge v \wedge z) \\
&\quad \vee (a \wedge y \wedge w) \vee (a \wedge y \wedge z) \\
&= u \vee (a \wedge x) \vee (a \wedge v \wedge z) \vee (a \wedge y \wedge w) \\
&= u \vee (a \wedge x) \vee (a \wedge b \wedge v \wedge z) \vee (a \wedge y \wedge w \wedge c) \\
&= u \vee (a \wedge x) \vee (a \wedge c \wedge v \wedge z) \vee (a \wedge y \wedge w \wedge b) \\
&\quad \leqslant u \vee (a \wedge x) \vee (c \wedge v \wedge z) \vee (y \wedge w \wedge b) \leqslant u \vee (a \wedge x) \\
&\quad \vee [v \wedge w \vee (v \wedge c \wedge z) \vee (b \wedge y \wedge w) \vee (b \wedge y \wedge c \wedge z)] \\
&= (u \vee (a \wedge x)) \vee \{[v \vee (b \wedge y)] \wedge [w \vee (c \wedge z)]\}.
\end{aligned}$$

The same inequality holds for the other components, hence M is modular.

Let $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 (0, 0, 1)$. Then $\alpha_i \vee \alpha_j = (1, 1, 1)$, $\alpha_i \wedge \alpha_j = (0, 0, 0)$ if $i \neq j$ and so $0, \alpha_1, \alpha_2, \alpha_3, 1$ from $\mathscr{M}_5$. For the remaining statements of the lemma we refer to [5].

COROLLARY 1. *The intervals* $[0, \alpha_1]$ *and* $[0, \alpha_2]$ *are projective; the corresponding unary algebraic function is* $p_{1,2}(x) = (x \vee \alpha_3) \wedge \alpha_2$, $[0, \alpha_1]$ *and* $[\alpha_1, 1]$ *are projective; the corresponding function is:*

$$q_1(x) = [(x \vee \alpha_3) \wedge \alpha_2] \vee \alpha_1 = p_{1,2}(x) \vee \alpha_1 .$$

Let us take two bounded lattices L_1 and L_2. Suppose that L_1 has a principal dual ideal $\mathscr{I}_1$, L_2 has a principal ideal $\mathscr{I}_2$ and $\mathscr{I}_1 \cong \mathscr{I}_2$. Let $\varphi: x \to x'$ denote this isomorphism. We can construct a lattice L as follows: L is the set of all $x \in L_1$ and $x \in L_2$, we identify x with x' for all $x \in \mathscr{I}_1$; $x \leqslant y$ have unchanged meaning if $x, y \in L_1$, or $x_1 y \in L_2$ and $x < y$ $x, y \notin \mathscr{I}_1 = \mathscr{I}_2$ iff $x \in L_1$, $y \in L_2$ and there exists a $z \in \mathscr{I}_1$ such that $x < z$ in L_1, and $z < y$ in L_2. Let $\vee_i$ denote the join operations in L_i, and let 0_2 denote the zero of L_2. It is easy to see that in L $\sup\{x, y\}$ always exists and

$$\sup\{x, y\} = \begin{cases} x \vee_i y & \text{if} \quad x, y \in L_i \\ (x \vee_1 0_2) \vee_2 y & \text{if} \quad x \in L_1 \quad \text{and} \quad y \in L_2 . \end{cases}$$

By the duality we have that L is a lattice.

We denote L as follows: $L = L_1 + L_2 (\varphi \mathscr{I}_1 = \mathscr{I}_2)$.

LEMMA 2. (Hall and Dilworth [3]). *Let L_1 be a principal ideal and L_2 be a principal dual ideal of the lattice L, such that $L_1 \cup L_2 = L$ and $L_1 \cap_2 \neq \phi$. If L_1 and L_2 are modular lattices then L is a modular lattice too. If Θ_1 and Θ_2 are congruence relations of L_1 resp. L_2 such that $\Theta_1|_{L_1 \cap L_2} = \Theta_2|_{L_1 \cap_2}$, then there exists a congruence relation Θ of L with the property $\Theta|_{L_i} = \Theta_i$ $(i = 1, 2)$.*

4. The construction

LEMMA 3. *Let K_0 be a modular lattice, which has the following properties:*

(1) *K_0 has a dual ideal $\mathscr{I}_0$ isomorphic to Q;*

(2) *every congruence relation of K_0 is the smallest extension of a congruence relation of $\mathscr{I}_0$.*

*Let $f: [a, b] \to [c, d]$, $a, b, c, d \in \mathscr{I}_0$ a *-operation on K_0, and let denote $K_0^* = \langle K_0; \vee, \wedge, f \rangle$. Then there exists a realization K of K_0^* which is a modular lattice and has a dual ideal $\mathscr{I}$ such that (1) and (2) are satisfied.*

Proof. Let us take the following two lattices:

I. Let S denote the lattice M given in Lemma 1 if we set $N = Q$. Then we take the intervals $S_i = [0, \alpha_i]$ and $S^i = [\alpha_i, 1]$ $i = 1, 2$ and the corresponding unary algebraic functions $p_{i,j}(x)$ $(i \neq j)$ and q_i given in the Corollary of Lemma 1.

II. $T = Q \times Q$. Let be $T_1 = [(0, 0), (1, 0)]$, $T_2 = [(0, 0), (0, 1)]$ and $T^1 = [(1, 0), (1, 1)]$, $T^2 = [(0, 1), (1, 1)]$. Then it is obviously $T_1 \cong T_2 \cong T^1 \cong Q$. T_1 and T^2 are projective; the unary algebraic function is $v_{1,2}(x) = x \vee (0, 1)$.

We define $A_0 = T + S(\psi T^1 = S_2)$ where φ is an arbitrary isomorphism between T^1 and S_2. Let A be the lattice $A_0 + T(\varphi S^1 = S_2)$ where ψ is again an arbitrary isomorphism (Fig. 2). Let σ denote the zero of A and let e_1, e_2, e_3, e_4 be the elements given in Fig. 2 (e_1 is the element $(1, 0)$, e_2 is the element $(1, 0)$ of the second T component of A; similarly e_3 is $(1, 0)$ and e_4 is $(0, 1)$ in the first T component of A).

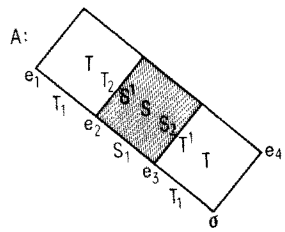

Fig. 2

By Lemma 1 and 2 we get: for every congruence relation Θ of $(e_1]$ (resp. $[e_1)$) there exists exactly one congruence relation $\bar{\Theta}$ of A such that for $x, y \in (e_1]$ (resp. $x, y \in [e_4)$) $x \equiv y(\bar{\Theta})$ iff $x \equiv y(\Theta)$. The intervals $[e_3, e_2]$ and $[e_1, e_2, \vee e_4]$ are projective, the corresponding unary function is $r_{21}q_1(x)$.

$(e_1]$ is the ordinal sum $T_1 + S_1 + T_1$ hence $(e_1]$ is isomorphic to Q. The same holds for the dual ideal $[e_4)$. Let now ϱ be an arbitrary isomorphism $\varrho : \mathscr{I}_0 \to (e_1]$ such that $\varrho(a) = e_3$ and $\varrho(b) = e_2$. Then we can define the lattice $B = K_0 + A(\varrho\mathscr{I}_0 = (e_1])$. (See Fig. 3.)

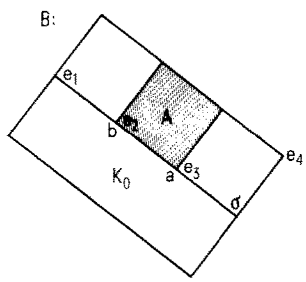

Fig. 3

The intervals $[a, b]$ and $[e_1, e_1 \vee e_4]$ are projective; we have namely the unary function $\lambda_1 = r_{21}q_1\varrho(x)$.

Take A in a second exemplar A' ($x \to x'$ let be an isomorphism between A and A'). If v is an isomorphism $[e_4) \to (e_1']$ such that $v(\varrho(c) \vee e_4) = e_3'$ and $v(\varrho(d) \vee e_4) = e_2'$ then we can define $C = B + A'(v(e_4] = [e_1'))$. We have the lattice on Fig. 4.

Using the Lemma 2 we get: for every congruence relation Θ of K_0 there exists exactly one congruence relation $\bar{\Theta}$ of C such that for $x, y \in K_0$ $x \equiv y(\bar{\Theta})$ iff $x \equiv y(\Theta)$. Similarly as in lattice B we have: $[e_1 \vee e_4, e_1 \vee e_4']$ and $[c, d]$ are projective; the corresponding function let be λ_2.

Finally let us take the lattice $D = S + S(\mu S^2 = S_1)$ with an arbitrary isomorphism μ (Fig. 5) $[d_2, d_1]$ and $[d_3, d_2]$ are projective; let λ_3 denote the corresponding unary function.

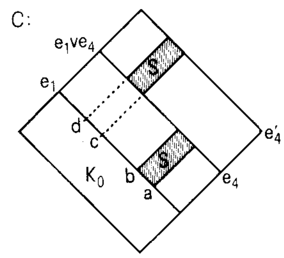

Fig. 4

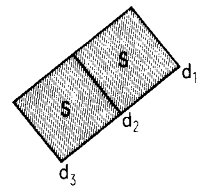

Fig. 5

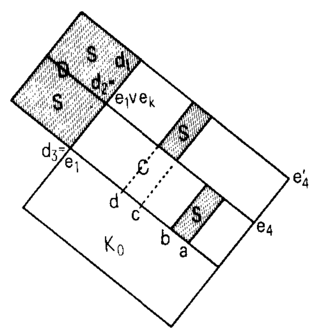

Fig. 6

The ideal $(d_1]$ is isomorphic to Q, hence it is isomorphic to the dual ideal $[e_1)$ of C. We shall define $K=C+D$ with an isomorphism $\varkappa:[e_1)\to(d]$ as follows: Let be $\lambda_2^{-1}\chi^{-1}\lambda_3\chi\lambda_1=f$ (Fig. 6). Then a congruence relation Θ of K_0 has an extension to K iff Θ is a congruence relation of K_0^* and every congruence relation of K is the extension of a congruence $\Theta\in\Theta(K_0)$. K is a realization of K_0^*. The dual ideal $\mathscr{I}$ generated by $e_4'\in K$ satisfies obviously (1) and (2). This proves Lemma 3.

To prove our theorem let be $K_0=Q_L=\langle Q;\vee,\wedge,f_1,f_2,\ldots\rangle$. Applying the Lemma 3 we get an extension K_1 of K_0 which realizes $\langle Q;\vee,\wedge,f_1\rangle$. We can extend K_1 to K_2 such that K_2 realizes $\langle Q;\vee,\wedge,f_1,f_2\rangle$. From K_i we can get on similar way K_{i+1}. By a direct limit procedure we get a lattice K which realizes Q_L and so $\mathbf{C}(K)\cong\mathbf{C}(Q_L)$. This proves $\mathbf{C}(K)\cong L$.

Remark 1. A modular lattice satisfying the identity

$$x\vee\bigwedge_{i=0}^{n}y_i=\bigwedge_{j=0}^{n}\Big(x\vee\bigwedge_{\substack{i=0\\ i\neq j}}^{n}y_i\Big)$$

is called an n-distributive lattice. (See G. Bergman [1] and A. Huhn [4].) A. Huhn has proved that L is n-distributive iff L does not contain a sublattice $B\cong 2^{n+1}$ and an element x such that x is the relative complement of all atoms of B in interval $[\inf B, \sup B]$. It is easy to show that the lattices T and S do not contain sublattice isomorphic to 2^3, therefore we have:

THEOREM 2. *Every finite distributive lattice is isomorphic to the congruence lattice of a 2-distributive lattice.*

Remark 2. Let K_0 be an arbitrary bounded distributive lattice and f_α $(\alpha\in\Omega)$ *-operations on K_0. In [6] it is proved[1]) that K_0 has a modular extension, which is a realization of $K_0^* = \langle K_0; \vee, \wedge, f_\alpha\rangle$.

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1) G. Grätzer proved the same statement for an arbitrary distributive lattice K_0 with zero.