On the Jordan—Dedekind Chain Condition.

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1. Introduction. The well known Jordan—Dedekind theorem of lattice theory was firstly generalised by G. BIRKHOFF ([1][1]) p. 66) who proved the following assertion.

Let L be a lattice satisfying the following two conditions[2]):

(α) $x \cap y \prec y$ implies $x \prec x \cup y$ $(x, y \in L)$;

(β) all bounded chains in L are finite.

Then

(JD) in L all maximal chains between fixed end points have the same length.

Some attempts have been made to get a more general form of this result. R. CROISOT [2] and G. SZÁSZ [3] proved that if we replace condition (β) by the weaker

(γ) there exists at least one finite maximal chain between a and b $(a < b;\ a, b \in L)$,

then it results that (JD) holds in the interval $[a, b]$. Although under weaker conditions, the Croisot—Szász theorem asserts the validity of (JD) only for the same family of lattices as the Birkhoff theorem. Therefore we have tried to generalise these theorems so that the general theorem be applicable to lattices with continuous as well as discrete chains.

We have also tried to obtain a statement analogous to condition (JD) in the case of infinite chains of arbitary power. We have shown that with a suitable definition of the length and the maximality of an infinite chain, in distributive lattices (JD) holds.

2. The case of finite chains. First we give a simplified proof[3]) for the

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1) Numbers in brackets refer to the Bibliography given at the end of this paper.

2) $a \prec b$ denotes that b covers a.

3) The idea of the proof is the same as of G. SZÁSZ [3].

Theorem 1 (The Croisot—Szász theorem). *Let L be a lattice satisfying (α), and C_1, C_2 two finite chains of L with the same end points. If C_1 is a (finite) maximal chain of length r, then*

(a) *C_2 is a finite chain;*

(b) *the length of C_2 is at most r;*

(c) *C_2 is maximal if and only if its length is r.*

Proof. Let

$$C_1 : a = a_0 < a_1 < \cdots < a_r = b.$$

We use an induction on r. The case $r = 1$ is trivial in any lattice. We assume the validity of the statement of the Theorem for $r-1$. Suppose it is possible to choose a subchain of C_2 of length $r+1$:

$$a = x_0 < x_1 < \cdots < x_{r+1} = b.$$

Consider the chain

$$(*) \qquad a_1 \leqq a_1 \cup x_1 \leqq \cdots \leqq a_1 \cup x_{r+1} = b$$

and denote by t the least integer with $x_t \geqq a_1$ $(t \geqq 1)$. If i and $i+1 \geqq t$, then trivially $a_1 \cup x_i < a_1 \cup x_{i+1}$. If i and $i+1 < t$, from $a \prec a_1$ it follows $a = x_i \cap a_1 = x_{i+1} \cap a_1$, hence in view of (α) $x_i \prec x_i \cup a_1$ and $x_{i+1} \prec x_{i+1} \cup a_1$, excluding the possibility $x_i \cup a_1 = x_{i+1} \cup a_1$. Consequently, $x_j \cup a_1 = x_{j+1} \cup a_1$ is impossible unless $j = t-1$. Thus the length of $(*)$ is r and the proof is completed.

We prove also the following, somewhat generalised form of the Croisot—Szász theorem.

Theorem 2. *Let L be a lattice satisfying (α), C_1 and C_2 two finite chains of L with the same end points. Then C_1 and C_2 can be refined so that the refined chains have the same length.*

Theorem 1 follows at once from Theorem 2. On the other hand, we show that Theorem 1 implies Theorem 2.

In the proof of the Theorem 2 we may assume, without loss of the generality, that the length m of C_2 is less than or equal to the length n of C_1. There exists a maximal chain [4]) M (with the same end points as C_1 and C_2) which is a refinement of C_2. If M has more than n elements, then C_2 has a refinement of length n and thus the statement of Theorem 2 is obvious. So we may suppose that M has at most n elements, but this contradicts Theorem 1.

[4]) The existence of M is equivalent to the Axiom of Choice of Zermelo ([1], pp. 42—43).

3. Counter-examples. G. SZÁSZ [4] proved that if we define the length of an infinite chain as the power of the set of its elements, and call an infinite chain maximal, if it is no proper subchain of any other one, then even in distributive lattices condition (JD) does not hold. This possibility is illustrated also by the following

Example 1. Let R be the chain of the rational numbers of the interval $[0, 1]$ and V the chain of all real numbers of $[0, 1]$. In the lattice $R \cdot V$ (i. e. in the cardinal product of R and V, in the sense of [1] p. 7) all the elements (x, x) (x rational) form a maximal chain between $(0, 0)$ and $(1, 1)$. This follows at once from the fact that in the case $y \neq z$, (y, z) and (x, x) are incomparable, where x is an arbitary rational number between y and z. Hence in $R \cdot V$ there exists a countable maximal chain between $(0, 0)$ and $(1, 1)$. On the other hand, the elements $(x, 0)$ and $(1, y)$ form a maximal chain of the power of continuum.

The following problem arises. Let C_1 and C_2 be maximal chains (with the same end points). Is then C_1 a homomorphic image of C_2 or C_2 a homomorphic image of C_1, at least in distributive lattices? In general, this assertion fails to hold as it is shown by the following

Example 2. Let A be a well-ordered and B a dually well-ordered infinite bounded chain with the bounds O_1, I_1 and O_2, I_2 $(O_1, I_1 \in A; O_2, I_2 \in B)$. In the lattice $A \cdot B$, all the elements (x, O_2) and (I_1, y) form a maximal chain C_1, and the elements (O_1, y) and (x, I_2) form a maximal chain C_2. Let us suppose e. g. that C_2 is a homomorphic image of C_1. Using the Duality Principle we may assume without loss of generality that the homomorphic image of (I_1, O_2) is greater than or equal to (O_1, I_2). In this case all the elements (O_1, y) of C_2 form a chain isomorphic with B, which is a convex subchain of the homomorphic image of A. Since a homomorphic image of a well-ordered chain is a well-ordered chain and a convex subchain of a well-ordered chain is again well-ordered, we get that B is a well-ordered and at the same time dually well-ordered chain, i. e. B is finite, in contradiction to the hypotheses.

4. The case of infinite chains. Our aim is to establish an analogon of the condition (JD) for infinite chains in distributive lattices. By a *cut* of a chain we mean a subdivision of the chain into two non-void convex subchains and define the length of a chain as the power of the set of its different cuts. Thus the length of a finite chain consisting of $n+1$ elements is n as usual, while e. g. the length of the chain of all rational numbers is equal to the power of the continuum.

A chain C well be called *strongly maximal,* if

(a) C is no proper subchain of any other one with the same end points;

(b) for every homomorphic image of C, (a) is valid.

With the aid of these notions we prove:

Theorem 3. *If L is a distributive lattice*[5]), *then all strongly maximal chains between fixed end points have the same length; i.e. an analogon of the condition (JD) holds in L.*

Proof. Let C be a strongly maximal chain in L with the end points a and b $(a<b)$. We cut C into two convex subchains, I and J $(a\in I, b\in J)$. We consider the congruence relation Θ of L induced by $I\equiv a$ and $J\equiv b$. In [5] we have shown the following assertion: If $z\notin[x,y]$, then $z\equiv x(\Theta_{x,y})$ is false ($\Theta_{x,y}$ denotes the congruence relation induced by $x\equiv y$). This result implies at once $a\not\equiv b(\Theta)$. Clearly from (b) $[a,b]/\Theta\simeq \mathbf{2}$[6]), hence Θ produces a cut on all chains between a and b. Θ is the minimal congruence relation with $a\equiv I$ and $J\equiv b$, but from $[a,b]/\Theta\simeq\mathbf{2}$ it is clear that Θ is the maximal one with the same property. It implies that in $[a,b]$ exists one and only one congruence relation with $I\equiv a$, $J\equiv b$ and $a\not\equiv b$, hence different congruence relations (which are induced by a cut of C) define different cuts on strongly maximal chains between a and b. Thus the length of a strongly maximal chain between a and b is equal to the power of the set of all congruence relations on $[a,b]$ which are induced by a cut of C. Thus the proof is completed[7]).

Remark. If, following A. G. Kuroš [6], we consider only complete chains, i. e. chains for which every cut goes through an element (i. e. either I has a l. u. b. or J has a g. l. b.), then the above notion of length coincides with the usual one. Since in a complete lattice every maximal chain is complete, we obtain that in complete distributive lattices Theorem 3 holds with the usual notion of length (but in general not with the usual notion of maximality).

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[5]) We conjecture that Theorem 3 holds in semi-modular lattices too.

[6]) $\mathbf{2}$ denotes the lattice with two elements.

[7]) We remark that it is possible that two chains have the same length, one of them is strongly maximal, the other has not this property. This may be shown by the two chains considered in Example 1.

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