On the definition of homomorphism kernels of lattices

Dedicated to JOSEF NAAS for his 60st birthday on October 16, 1966

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The usual definition of homomorphism kernels of lattices is as follows: Let L and L' be two lattices such that L' has a 0 element. If Φ is a homomorphism of the lattice L onto L' then the set of all $x \in L$ for which $x\,\Phi = 0$ forms the kernel of the homomorphism Φ. This definition – using a familiar logical expression – is of second order type, and so this fact naturally raises the question whether the notion of homomorphism kernel can be characterized in first order terms? This means precisely the following. Does a formula $\mathcal{F}$ of the first order logic with identity exist, such that $\mathcal{F}$ contains as primitiv non logical constants only the lattice operations and a symbol A for a subset of the universe and such that $\mathcal{F}$ is true in a lattice L with a specified subset L' as the interpretation for A if and only if L' is a homomorphism kernel of L? Our theorem proves that such a universal first order formula does not exist (a universal first order formula is formed from an open formula by prefixing to it universal quantifiers binding all its variables).

Theorem A. *The notion of homomorphism kernels of lattices can not be characterized by a universal first order formula.*

This theorem is obviously implied by the following:

Theorem B. *Let n be a natural number. There exists a lattice L an an ideal I of L which is not a homomorphism kernel such that for every* $y_1, y_2, \ldots, y_n \in L$ *in the sublattice generated by* $y_1, y_2, \ldots, y_n$ *and I the ideal I is a homomorphism kernel.*

Proof. We have to construct the lattice L. Let n be a natural number. Consider the chain C of the length $n + 2$. The elements of C are denoted by $0, 1, 2, \ldots, n + 2$. Take the direct product $D = C \times C$. The elements of this lattice are of the form (s, t) where $0 \leqq s \leqq n + 2$ and $0 \leqq t \leqq n + 2$. Further, we define new elements $x_k (k = 0, 1, \ldots, 2\,n + 1)$ satisfying the following relations:

$$\left.\begin{aligned}
x_{2r} &\frown (r+2, r+1) = x_{2r} \frown (r+1, r+2) = (r+1, r+1),\\
x_{2r} &\smile (r+2, r+1) = x_{2r} \smile (r+1, r+2) = (r+2, r+2),\\
x_{2r-1} &\frown (r+1, r+1) = x_{2r-1} \frown (r, r+2) = (r, r+1),\\
x_{2r-1} &\smile (r+1, r+1) = x_{2r-1} \smile (r, r+2) = (r+1, r+2),
\end{aligned}\right\} r = 1, 2, \ldots, n$$

and for x_0 and x_{2n+1} we have:

$$\begin{aligned}
x_0 &\frown (2, 0) = x_0 \frown (1, 1) = (1, 0),\\
x_0 &\smile (2, 0) = x_0 \smile (1, 1) = (2, 1),\\
x_{2n+1} &\frown (1, n+1) = x_{2n+1} \frown (0, n+2) = (0, n+1),\\
x_{2n+1} &\smile (1, n+1) = x_{2n+1} \smile (0, n+2) = (1, n+2).
\end{aligned}$$

Thus we have got a lattice $L = D \vee \{x_0, \ldots, x_{2n+1}\}$. We define the ideal I as the subset $\{(0,0), (0,1)\}$. Fig. 1 helps to visualize the construction for $n = 3$.

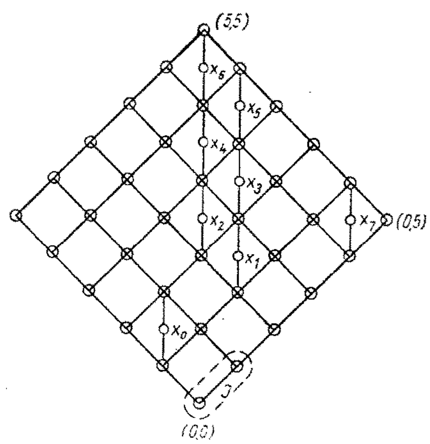

Fig 1.

By the definition of the elements x_i we obtain that the sublattices:

$D_0 = \{(1,0), x_0, (2,0), (1,1), (2,1)\}$,

$D_{2r} = \{(r+1, r+1), x_{2r}, (r+2, r+1), (r+1, r+2), (r+2, r+2)\}$,

$D_{2r-1} = \{(r, r+1), x_{2r-1}, (r+1, r+1), (r, r+2), (r+1, r+2)\}$,

$D_{2n+1} = \{(0, n+1), x_{2n+1}, (1, n+1), (0, n+2), (1, n+2)\}$

are isomorphic to the modular non distributive lattice containing five elements. This lattice is a simple one i. e. it has no proper congruence relations.

First we prove that I is not a homomorphism kernel in L. Suppose on the contrary, that is a homomorphism kernel; then there exists a congruence relation Θ of L such that I is a class by Θ, and so $(0,0) \equiv (0,1)$ (Θ), thus

$$(1,0) = (1,0) \cup (0,0) \equiv (1,0) \cup (0,1) = (1,1) \ (\Theta).$$

But D_0 is a simple lattice which insolves $(1,1) \equiv (2,1)$ (Θ). So we get

$$(1,2) = (1,1) \cup (1,2) \equiv (2,1) \cup (1,2) = (2,2) \ (\Theta).$$

But $(1,2)$, $(2,2) \in D_1$. Hence $(2,2) \equiv (2,3)$ (Θ). On the same way we get that the elements of every D_i are congruent by Θ i. e.

$$(0, n+1) \equiv (1, n+1) \ (\Theta)$$

consequently $(0,0) = (0, n+1) \cap (1,0) \equiv (1, n+1) \cap (1,0) = (1,0)$ (Θ) but $(0,0) \in I$, $(1,0) \notin I$ which is a contradiction to our supposition on I.

It remains to prove that for arbitrary elements $y_1, y_2, \ldots, y_n$ the ideal I is a homomorphism kernel in the sublattice generated by $y_1, y_2, \ldots, y_n$ and I. Let us denote $L' = \{y_1, y_2, \ldots, y_n, I\}$.

The number of the $x_i - s$ is $2n + 2$ and so we have that there exists an x_i $(0 \leqq i \leqq 2n + 1)$ different from all $y_j - s$. But x_i is an irreducible element ($\cup$ and $\cap$ -irreducible) in L and so $x_i \notin L'$. Let

$$L_i'' = L \setminus \{x_i\} \quad (i = 0, \ldots, 2n + 1).$$

It is obvious that $L_i'' \supseteqq L'$. If we prove that I is a homomorphism kernel in L_i'' for all i then this is true in L' too. We must distinguish four cases:

I. $i = 0$, i. e. $x_0 \notin L'$. Then we define Θ_0 on L_0''. This is the following (Fig. 2) $a \equiv b(\Theta_0)$ $(a < b,\ a, b \in L_0'')$ if an only if

$$a = (t, 0), \quad b = (t, 1) \quad (t = 0, 1, \ldots, n + 2).$$

It is routine to check that Θ_0 is a congruence relation.

II. $i = 2r - 1$ $(r = 1, 2, \ldots, n)$. Then we define Θ_i on L_i'' as follows $a \equiv b\ (\Theta_i)$ $(a < b, a, b \in L_i'')$ (Fig. 3) if and only if one of the following conditions hold:

α. $a = (t, 2s)$, $b = (t, 2s + 1)$ $(t = 0, 1, \ldots, n + 2)$
$(s = 0, 1, \ldots, r)$;

β. $(1, u) \leqq a$, $b \leqq (r, u)$ $(u = 0, 1, \ldots, n + 2)$;

γ. there exists a c, $a < c < b$ such that $a \equiv c$ under α, and $c \equiv b$ under β.

III. $i = 2r$ $(r = 1, 2, \ldots, n)$. Then $a \equiv b(\Theta_i)$ $(a < b, a, b \in L_i'')$ if and only if one of the following conditions are satisfied:

α. $a = (t, 0)$, $b = (t, 1)$ $(t = 0, 1, \ldots, n + 2)$;

β. $(t, 2) \leqq a$, $b \leqq (t, 2r)$ $(t = 0, 1, \ldots, n + 2)$;

γ. $(1, 0) \leqq a$, $b \leqq (r + 1, 1)$;

δ. $(1, 2) \leqq a$, $b \leqq (r + 1, r + 2)$;

ε. $(1, t) \leqq a$, $b \leqq (r + 1, t)$ $(t = 0, 1, \ldots, n + 2)$.

Θ_i is obviously a congruence relation (Fig. 4).

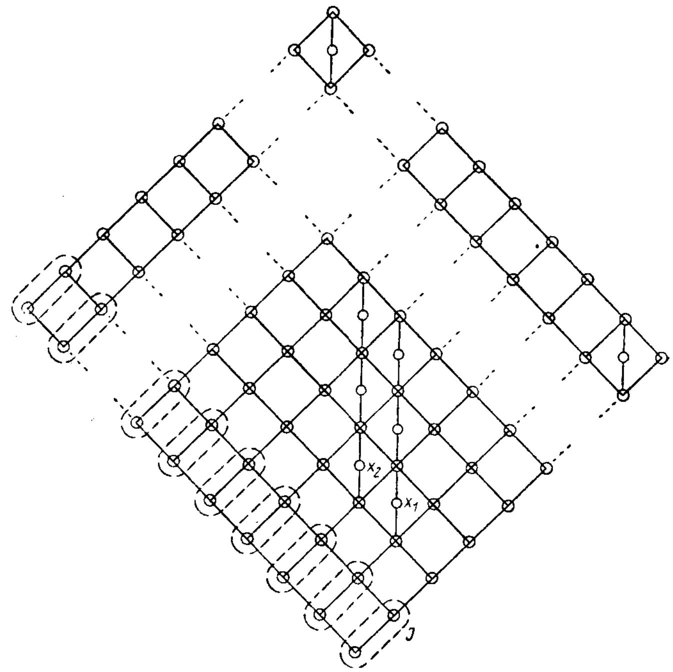

Fig. 2

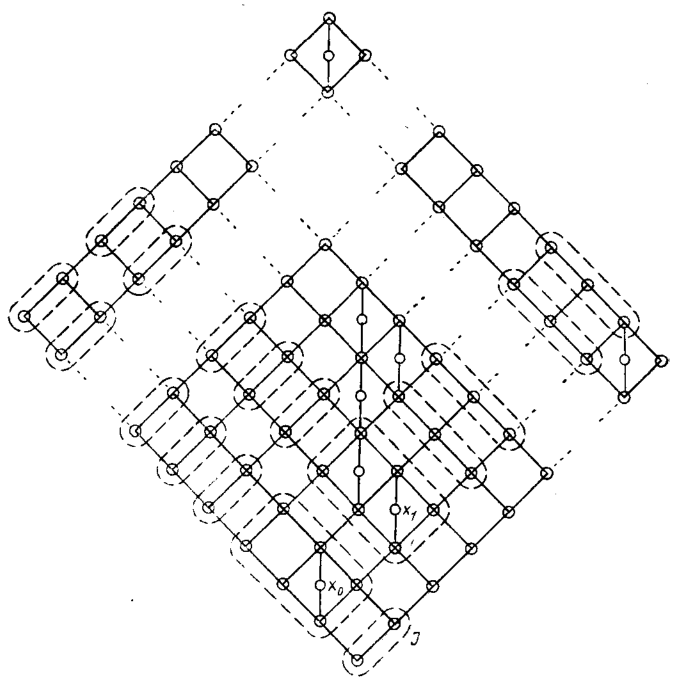

Fig. 3

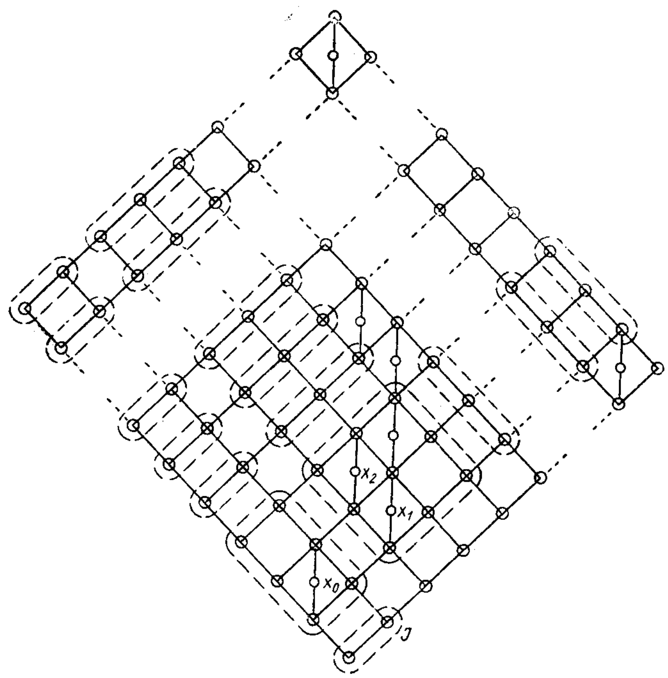

Fig. 4

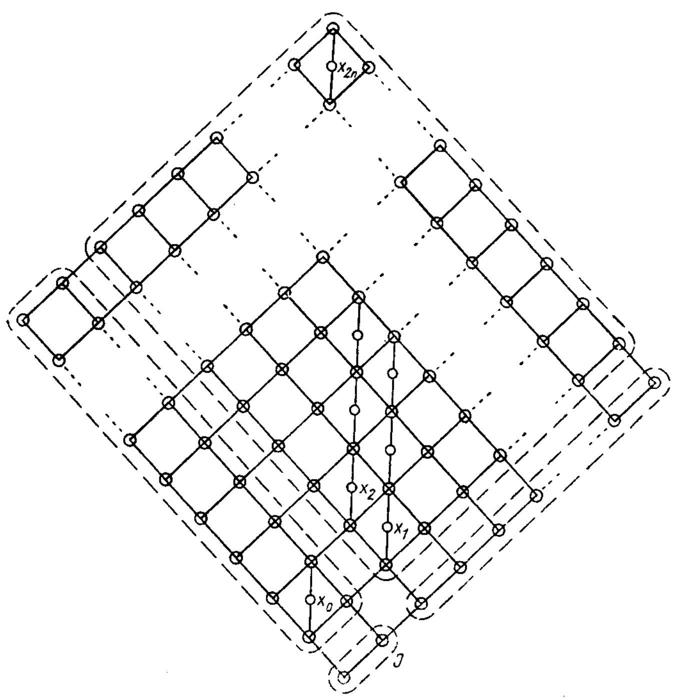

Fig. 5

IV. Finally we define Θ_{2n+1} on L''_{2n+1}:

$$a \equiv b(\Theta_{2n+1}) \quad (a > b)$$

if and only if

α. $a, b \geqq (1, 2)$;
β. $(n + 2, 1) \geqq a, \quad b \geqq (1, 0)$;
γ. $(0, n + 2) \geqq a, \quad b \geqq (0,2)$;
δ. $a, b \in I$.

Q. e. d.

We note that lattice L is a modular lattice, and so we get that in the class of all modular lattices the notion of homomorphism kernel can not be characterized by a universal first order formula. In distributive lattices every ideal is a homomorphism kernel and so in the class of all distributive lattices we can characterize the homomorphism kernel in first order terms.