A NOTE ON A SPECIAL TYPE OF FULLY INVARIANT SUBGROUPS OF ABELIAN GROUPS

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To the memory of Professor L. FEJÉR

In this note group always means abelian group and G stands for a group.

In the theory of groups it is a general phenomenon that the direct summand property of a subgroup A of G is proved in the following way: we take a well-defined subgroup A^* of G for which $A \cap A^* = O$ and then a subgroup M which is maximal with respect to the properties $A^* \subseteq M$ and $A \cap M = 0$ and afterwards we try to prove somehow that $G = A + M$. Mostly, the subgroup A^* is fully invariant.

An example is a theorem of SZELE asserting that if $A = \sum C(p^k)$ and $A \cap p^k G = O$, then A is a direct summand of G. In this case[1] $A^* = p^k G$. Another example is a theorem of ERDÉLYI[2].

This suggests that to every direct summand A of G there corresponds a greatest fully invariant subgroup A^* which is contained in every complement of A. This feeling is strengthened by the following theorem of L. FUCHS[3]: $G = A + B = A + B_1$ implies $B = B_1$ if and only if B is fully invariant, i. e. in case B is the A^* itself.

The idea of the proof of this theorem leads to the existence of A^* in the general case:

THEOREM 1. *Let A be a direct summand of G and $\{B_\lambda\}$ $(\lambda \in \Lambda)$ the set of all complements of A $(G = A + B_\lambda$ for every $\lambda \in \Lambda)$. Then $A^* = \bigwedge_{\lambda \in \Lambda} B_\lambda$ is the greatest fully invariant subgroup of G satisfying $A \cap A^* = O$, i. e. if F is fully invariant and $A \cap F = O$, then $F \subseteq A^*$.*

Of course, the above mentioned theorem of L. Fuchs is immediate from Theorem 1.

It is natural to ask whether in case $A = \sum C(p^k)$ the equality $A^* = p^k G$ holds or not. It is easy to see that the answer is in the affirmative. In general, if A is a bounded p-group we can always determine A^*:

THEOREM 2. *Let A be a direct summand of G and suppose A is a bounded p-group with the minimal bound p^k (i. e. $p^k A = O$ but $p^{k-1} A \neq O$). Then the meet A^* of all complements of A equals $p^k G$.*

If A is an unbounded p-group, then we are able to describe A^* only under an additional hypothesis:

[1] See e.g. in the book of L. FUCHS, *Abelian groups*, (Budapest, 1958), p. 79.
[2] Ibid. p. 81.
[3] Ibid. p. 76.

Theorem 3. *Let G be a reduced torsion group and A a direct summand of G such that A is an unbounded p-group. Then*[4] $A^* = \sum_{q \neq p} G_q$.

Corollary. Under the hypotheses of Theorem 3, the complement of A in G is uniquely determined if and only if $A = G_p$.

In the two examples given above, any M, which is maximal with respect to the properties $M \supseteq A^*$ and $A \cap M = O$, was a complement of A. This does not hold in general. A necessary and sufficient condition for this to hold is the content of the following assertion which is but a trivial consequence of a result of L. Fuchs[5].

Theorem 4. *Let A be a direct summand of G and A^* as defined in Theorem 1. Any M containing A^* and maximal with respect to $A \cap M = O$ is a complement of A if and only if one of the following conditions is satisfied:*

α) *A is divisible;*

β) *$G/_{A+A^*}$ is a torsion group and $p^t(G/_{A+A^*})_p = O$, whenever there exists an element in A not in pA of order p^t.*

Finally, we mention that in Szele's theorem, mentioned at the beginning, A may not be replaced by a bounded p-group of any other type, that is, Szele's result is the best possible one. Because if A is a bounded p-group with the bound p^k *then A is a direct summand of every containing group G with $A \cap p^kG = O$ when and only when* $A = \sum C(p^k)$.

For the notations and terminology we refer to the book of L. Fuchs, cited in footnote 1.

Proof of the results. For the proof of Theorem 1 we need a lemma.

Lemma 1. Let $G = A + B$ and H a subgroup of B. If φ is an endomorphism of G such that $H\varphi \nsubseteq B$, then there exists a complement B_1 of A not containing H.

Proof. Let η and Θ be the projections[6] of G onto A and B. We may suppose $A\varphi = O$. Define $\eta_1 = \eta + \varphi\eta$ and $\Theta_1 = \Theta - \varphi\eta$. It is routine to check that η_1 and Θ_1 are projections satisfying $\eta_1\Theta_1 = O$, $\eta_1 + \Theta_1 = \iota$, thus $G = G\eta_1 + G\Theta_1$ and $G\eta_1 = A$. We prove that $H \nsubseteq G\Theta_1$. Suppose that H is contained $G\Theta_1$ and let $h \in H$ such that $h\varphi \notin B$. $h\varphi = a + b\,(a \in A,\ b \in B)$, $a \neq O$, thus $h\Theta_1 = h\Theta - (h\varphi)\eta = h - a$ and $h\Theta_1 \in G\Theta_1$. Now, from $h \in G\Theta_1$ it follows $a = h - (h - a) \in G\Theta_1$, contradicting $A \cap G\Theta_1 = O$. Thus $H \nsubseteq G\Theta_1$ and so $B_1 = G\Theta_1$ is a desired complement of A.

Now it is easy to prove Theorem 1. Indeed, put $A^* = \bigwedge B_\lambda$, then from Lemma 1 it follows that A^* is a fully invariant subgroup. Further, if H is a fully invariant subgroup with $A \cap H = O$ then[7] $H \cap (A + B_\lambda) = (H \cap A) + (H \cap B_\lambda) = H \cap B_\lambda$, thus $H \subseteq B_\lambda$ for all $\lambda \in \Lambda$ and so $H \subseteq A^*$, finishing the proof of Theorem 1.

The proofs of Theorems 2 and 3 are based on

[4] p, q denote prime numbers; G_p is the p-component of G.

[5] Ibid. p. 75.

[6] A projection is an idempotent endomorphism. ι is the identity automorphism. If η and Θ are endomorphisms then $\eta + \Theta$ and $\eta\Theta$ are defined as usual: $x(\eta + \Theta) = x\eta + x\Theta$ and $x(\eta\Theta) = (x\eta)\Theta$.

[7] See ibid. p. 72.

LEMMA 2. Let $G = A + B$, $a \in A$ and $b \in B$ elements of order p and[8] $\infty > H_p(a) \geq H_p(b)$. Then there is an endomorphism φ of G such that $b\varphi = a$.

PROOF. By a theorem of KULIKOV[9] there exist decompositions $A = \{a_1\} + A_1$ and $B = \{b_1\} + B_1$ such that $a \in \{a_1\}$ and $b \in \{b_1\}$, further, the condition $H_p(a) \geq H_p(b)$ implies $o(a_1) \geq o(b_1)$. It follows the existence of an isomorphism ψ of $\{b_1\}$ into $\{a_1\}$ such that $b\psi = a$.

We define an endomorphism φ by the rules: $\{a_1\}\varphi = A_1\varphi = B_1\varphi = O$ and $x\varphi = x\psi$ for $x \in \{b_1\}$. Obviously, φ satisfies the requirements.

We need also the following — in the literature frequently used —

LEMMA 3. Let $x \to \bar{x}$ be a homomorphism of G onto $\bar{G}$ with the kernel K and A a subgroup of G such that $A \cap K = O$. $\bar{G} = \bar{A} + \bar{B}$ if and only if $G = A + B$ with $B \supseteq K$.

Now we prove Theorem 2. If $G = A + B$, then $p^kG = p^kA + p^kB = p^kB$, thus $p^kG \subseteq B$, whence $p^kG \subseteq A^*$. Thus from Lemma 3 we get that it is enough to prove in G/p^kG that $A^* = O$. It is the same as to say that we may suppose $p^kG = O$. If $A^* \neq O$, then there exists $O \neq b \in A^*$ of order p. Since the bound of G is that of A, it follows the existence of an element $a \in A$ b of order p such that $\infty > H(a) \geq H(b)$. Applying Lemma 2 we get the existence of an endomorphism φ such that $b\varphi = a$, contradicting $A^*\varphi \subseteq A^*$ (Theorem 1) and $A \cap A^* = O$. Thus Theorem 2 is proved.

Theorem 3 may be proved quite analogously, first reducing to the case of p-groups, using the fact that in case A is a p-group, $G = A + B$, and $p \neq q$, then $G_q = A_q + B_q = B_q \subseteq B$, thus $\sum_{q \neq p} G_q \subseteq A^*$ and then considering $G/\sum_{q \neq p} G_q$ instead of G. Then we argue as above, mutatis mutandis.

Theorem 4 does not need a proof, only the observation that owing to Lemma 3 we may discuss the case $A^* = O$ in which case Theorem 4 is reduced to Fuchs's theorem.

Finally, we prove the italicized assertion, stated after Theorem 4. Let A be a bounded p-group with the bound p^k and of rank $\mathfrak{m}$ not of the type $\sum C(p^k)$. Then $A = \sum \{a_\alpha\}$, $o(a_\alpha) \leq p^k$. We may imbed A in $G = \sum_{\mathfrak{m}} C(p^k)$ in the natural way. $A \cap p^kG = O$ holds (for $p^kG = O$), but A is not a direct summand of G, for the set of elements of order p is the same in A as in G.

[8] $H_p(x)$ denotes the height of the element x at the prime p.
[9] See ibid p. 80.