SEPARATUM

ACTA MATHEMATICA
ACADEMIAE SCIENTIARUM HUNGARICAE

TOMUS XIII FASCICULI 1—2

G. GRÄTZER and E. T. SCHMIDT

# ON CONGRUENCE LATTICES OF LATTICES

1962

# ON CONGRUENCE LATTICES OF LATTICES

By

G. GRÄTZER and E. T. SCHMIDT (Budapest)

(*Presented by* L. RÉDEI)

In this note we prove some theorems on congruence lattices of lattices. The results are listed in § 1, where two unsolved problems are also mentioned. The proofs are given in §§ 2–3.

## § 1. Results and problems

If $K$ is a lattice, then let $\Theta(K)$ denote the lattice of all congruence relations of $K$. It is known (see [1]) that $\Theta(K)$ is a distributive lattice satisfying some continuity properties (see below). It is natural to ask about the lattice-theoretical characterization of $\Theta(K)$. If $K$ is finite, then $\Theta(K)$ is also finite, and conversely, every finite distributive lattice $L$ is isomorphic to a $\Theta(K)$ where $K$ is finite too. This theorem is due to R. P. DILWORTH and is mentioned in [1] without proof. No proof of this theorem has been published as yet.

In this note we give a proof of this theorem; some generalizations are also mentioned.

Before stating the results some notions are needed. A lattice $K$ is called section complemented if $K$ has a least element 0, and if every $x$ with $x \leqq y$ has a complement $z$ in $[0, y]$, i. e. $x \cap z = 0$, $x \cup z = y$. The length of a chain $C$ of $n+1$ elements is $n$, and the length of a finite lattice $K$ is $n$ if $K$ contains a subchain of length $n$ but no subchain of length $n+1$.

THEOREM 1. *Let $L$ be a finite distributive lattice of length $n$. Then there exists a finite lattice $K = K(L)$ having the following properties:*

(i) *$K$ is section complemented;*
(ii) *$\Theta(K)$ is isomorphic to $L$;*
(iii) *the length of $K$ is at most $2n-1$;*
(iv) *if $L$ is irreducible, then also $K$ is irreducible*[1];
(v) *the congruences of $K$ are permutable.*

We do not use[2] the finiteness of $L$, only the fact that the partially ordered set $P$ of join irreducible elements of $L$ determines $L$, in fact: $L \cong 2^{\tilde{P}}$ ($\tilde{P}$ denotes the dual of $\tilde{P}$, $2^{\tilde{P}}$ denotes the lattice of all monotone functions defined on $\tilde{P}$ with values in the chain **2** of two elements). Thus we get

[1] In fact, much more is true. If $K_1 = K(L_1)$, $K_2 = K(L_2)$, $L = L_1 \times L_2$, $K = K(L)$, then $K = K_1 \times K_2$ and, conversely, if $K = K(L)$, $K = K_1 \times K_2$, then we can decompose $L = L_1 \times L_2$ so that $K_1 = K(L_1)$, $K_2 = K(L_2)$.

[2] This remark is also due to R. P. DILWORTH.

THEOREM 2. *Let $P$ be a partially ordered set. Then there exists a section complemented locally finite*[3] *lattice $K$ such that $\Theta(K) \cong 2^P$.*

One can give several lattice-theoretical characterizations of $2^P$. Some of them are included in

THEOREM 3. *The following conditions*[4] *on the lattice $L$ are equivalent:*

(i) *there exists a partially ordered set $P$ such that $L \cong 2^P$;*

(ii) *$L$ is isomorphic to a complete sublattice of an atomic complete Boolean algebra;*

(iii) *$L$ is complete and $\bigvee$-distributive and every element of $L$ is a (complete) join of completely join-irreducible elements;*

(iv) *$L$ is a distributive compactly generated lattice in which every compact element is a finite join of join-irreducible compact elements;*

(v) *$L$ is isomorphic to the lattice of all ideals of $F$, where $F$ is a distributive join semilattice with zero, such that every element of $F$ is a finite join of join-irreducible elements.*

Finally, we give some conditions on a lattice $K$ assuring that $\Theta(K)$ satisfies the conditions of Theorem 3.

If $K$ is a lattice, and $a, b, c, d \in K$, then we write $\overline{a, b} \xrightarrow{1} \overline{c, d}$ if

$$(a \cap b) \cup (c \cap d) = c \cap d, \quad (a \cup b) \cup (c \cap d) = c \cup d$$

or

$$(a \cup b) \cap (c \cup d) = c \cup d, \quad (a \cap b) \cap (c \cup d) = c \cap d.$$

$\overline{a, b} \rightarrow \overline{c, d}$ means the existence of a sequence $x_1, y_1, \dots, x_n, y_n$ such that $\overline{a, b} \xrightarrow{1} \overline{x_1 y_1} \xrightarrow{1} \dots \xrightarrow{1} \overline{x_n y_n} \xrightarrow{1} \overline{c, d}$. The interval $[a, b]$ is *irreducible* if $a = z_0, \dots, z_n = b$, and $\overline{e, f} \rightarrow \overline{z_{i-1}, z_i}$ or $\overline{g, h} \rightarrow \overline{z_{i-1}, z_i}$ for every $i$ imply that either the first or the second relation holds for all $i$. Following CRAWLEY [4], $[a, b]$ is called *minimal* if $\overline{a, b} \rightarrow \overline{e, f}$ implies the existence of a sequence $a = z_0, \dots, z_n = b$, such that $\overline{e, f} \rightarrow \overline{z_{i-1}, z_i}$ for all $i$. Obviously, every minimal interval is irreducible, the converse does not hold in general.

THEOREM 4. *The following conditions on the lattice $K$ are equivalent:*

(1) *for any $a, b \in K$ $(a \leqq b)$ there is a sequence $a = z_0, \dots, z_n = b$ such that all the intervals $[z_{i-1}, z_i]$ are irreducible;*

(2) *in $\Theta(K)$ the law (DID) $\Theta \cup \bigwedge(\Theta_\alpha, \alpha \in A) = \bigwedge(\Theta \cup \Theta_\alpha; \alpha \in A)$ unrestrictedly holds;*

(3) *there exists a partially ordered set $P$ such that $\Theta(K) \cong 2^P$;*

(4) *$\Theta(K)$ is isomorphic to a complete sublattice of an atomic complete Boolean algebra.*

[3] I. e. every interval $[0, a]$ is finite.

[4] $x \in L$ is join irreducible if $x = \bigvee_{i=1}^{n} y_i$ implies $x = y_i$ for some $i = 1, 2, \dots, n$; completely join irreducible if $x = \bigvee(y_\alpha, \alpha \in A)$ implies $x \in \{y_\alpha, \alpha \in A\}$. The element $x$ is called compact if $x \leqq \bigvee(y_\alpha, \alpha \in A)$ implies $x \leqq \bigvee(y_\alpha, \alpha \in B)$ for some finite $B \subseteq A$. $L$ is compactly generated if it is complete and every element is the (complete) join of compact elements. A join semilattice $F$ is distributive if $t \leqq x \cup y$ implies $t = x_1 \cup y_1$ with $x_1 \leqq x$, $y_1 \leqq y$. A non-void subset $I$ of $F$ is an ideal if $x \cup y \in I$ is equivalent to $x, y \in I$. The set $I(F)$ of all ideals of a distributive join semilattice $F$ partially ordered under set inclusion is a distributive lattice. $L$ is $\bigvee$-distributive if $x \cap \bigvee(x_\alpha, \alpha \in A) = \bigvee(x \cap x_\alpha, \alpha \in A)$.

Condition (1) is rather complicated, the equivalence of the others is more interesting. We included (1) in order to get a theorem of CRAWLEY as a simple corollary. Some easy corollaries of Theorem 4 are the following:

COROLLARY 1. (CRAWLEY [4].) *$\Theta(K)$ is a Boolean algebra if and only if for any $a, b \in K$ ($a \leqq b$) there is a sequence $a = z_0, \dots, z_n = b$ such that all the $[z_{i-1}, z_i]$ are minimal.*

COROLLARY 2. (HASHIMOTO [8], GRÄTZER and SCHMIDT [5].) *The following conditions on a distributive lattice $K$ are equivalent:*

(1) *$K$ is locally finite (i. e. every interval is finite);*

(2) *$\Theta(K)$ is a Boolean algebra;*

(3) *in $\Theta(K)$ (DID) unrestrictedly holds.*

COROLLARY 3. *$\Theta(K)$ is a chain if and only if every interval is irreducible.*

The following two problems are worth mentioning:

PROBLEM 1. Let $L$ be a compactly generated distributive lattice. Does there exist a lattice $K$ such that $\Theta(K) \cong L$. Are further conditions on $L$ necessary if we require $K$ to be section complemented?

PROBLEM 2. Determine the least integer $\delta(n)$ such that to any distributive lattice $L$ of length $n$ there exists a lattice $K$ with $\Theta(K) \cong L$ and of length at most $\delta(n)$.

## § 2. The proof of Theorem 1

Let $L$ be a distributive lattice, and $P = \{p, q, r, \dots\}$ the set of non-zero join irreducible elements of $L$. The partial ordering relation in $P$ is denoted by $<$ the covering relation by $\prec$. Our goal is to construct a lattice $K$ with the properties (i)–(v) of Theorem 1.

We define the set $H$ as follows: the elements of $H$ are those of $P$ taken in two copies: $q^1, q^2$ ($q \in P$); we set $q^1 = q^2$ if and only if $q$ is maximal in $P$. Let us agree that $q'$ denotes any one of $q^1, q^2$; then $q''$ will stand for the other of $q^1, q^2$.

We say that a subset $G$ of $H$ is closed if

(1) $p \prec q$ and $q', p' \in G$ imply $p'' \in G$. It is trivial that the set $H$ is closed, and the intersection of any number of closed sets is again closed. Thus the closed subsets of $H$ form a lattice $K = K(L)$. We prove that $K$ satisfies (i)–(v) of Theorem 1.

If $G \subseteq H$, there exists a least closed set $\bar{G}$ containing $G$. We denote by $\cup$ and $\cap$ the join and meet in $K$, while by $\vee$, $\wedge$, $\setminus$ the set theoretical join, meet and difference. We identify the element $p'$ of $H$ with the atom $\{p'\}$ of $K$. If $G, N \in K$, then

$$G \cup N = \overline{G \vee N}, \quad G \cap N = G \wedge N.$$

Now let $G, N \in K$, $G \supseteq N$. We define $F$ as $(G \setminus N) \setminus F_1$ where $F_1$ consists of all $p' \in G \setminus N$ satisfying

(2) there exists a $q$ such that $p \prec q$, $q' \in G \setminus N$, $p'' \in N$.

We prove that $F$ is the complement of $N$ in $G$. First we prove $F \in K$, i. e. that $F$ satisfies (1). Let us suppose $p \prec q$, $q', p' \in F$. $G$ is closed, thus $p'' \in G$. But $p'' \in N$

is impossible, for this implies $p' \in F_1$ by (2), contradicting $p' \in F$. $p'' \in F_1$ implies by (2) $p' \in N$, contradicting $p' \in F$, hence $p'' \notin F_1$. Thus $p'' \in G$, $p'' \notin N$, $p'' \notin F_1$, therefore $p'' \in F$, so $F$ is closed.

$F \cap N \subseteq (G \setminus N) \wedge N = 0$, therefore $F \cap N = 0$. Finally, we prove $F \cup N = G$. But $F \vee N \subseteq (G \setminus N) \vee N = G$, hence it is enough to prove that $p' \in G$ implies $p' \in F \cup N$. But if $p' \in G$, $p' \notin F \cup N$, then $p' \in F_1$, hence $p'$ and a suitable $q$ satisfy (2). We choose $p$ so that $q$ be as great as possible. $q' \in F$, for if $q' \notin F$, then $q' \in F_1$ and there exists an $r$ such that $q \prec r$, $r' \in G \setminus N$. Thus $q'$ and $r'$ satisfy (2), $q \prec r$ contradicting the maximality of $q$. Thus $q' \in F$, $p' \in F_1$, and so $p'' \in N$. Hence $p \prec q$, $p''$, $q' \in F \cup N$, so (1) implies $p' \in F \cup N$ finishing the proof of part (i) of Theorem 1.

Now we fix $r \in P$ and define $A(r)$ to consist of all $p', p''$ such that $p \leqq r$. $A(r)$ obviously satisfies

(3) $p' \in A(r)$ implies $p'' \in A(r)$.

Thus every $A(r)$ satisfies (1) i. e. $A(r) \in K$. Now we prove the equality

(4) $A(r) \cup N = A(r) \vee N$ for every $N \in K$.

It is enough to prove that $A(r) \vee N$ is closed. To this end let $p'', q' \in A(r) \vee N$ and $p \prec q$. If any one of $p''$, $q'$ is in $A(r)$, then $p \leqq r$, thus $p' \in A(r) \vee N$. If no one of them is in $A(r)$, then $p'', q' \in N$, thus by the closedness of $N$ we get $p' \in N \subseteq A(r) \vee N$, so $A(r) \vee N$ is closed.

The equality

$$A(r) \vee (X \wedge Y) = (A(r) \vee X) \wedge (A(r) \vee Y) \quad (X, Y \in K)$$

is trivial. But every $\vee$ may be replaced by $\cup$ owing to (4) and the $\wedge$ by $\cap$, so we get

$$A(r) \cup (X \cap Y) = (A(r) \cup X) \cap (A(r) \cup Y) \quad \text{for all} \quad X, Y \in K,$$

which means, in the terminology of O. Ore [11], that $A(r)$ is a distributive element of $K$, implying ([11], pp. 622–623) that the principal ideal generated by $A(r)$ is a congruence class under a suitable congruence relation.

It is well known ([1], p. 23) that in a section complemented lattice every congruence relation $\Theta$ is completely determined by the ideal $I(\Theta) = \{x; x \equiv 0\ (\Theta)\}$. Thus $\Theta > \Phi$ if and only if $I(\Theta) \supset I(\Phi)$. Further, every ideal $I$ of a finite lattice is determined by its greatest element; let $A(\Theta)$ denote the greatest element of $I(\Theta)$. Thus $\Theta \to A(\Theta)$ is a one-to-one order preserving correspondence between $\Theta(K)$ and the elements $A(\Theta)$.

We have already proved that every $A(r)$ is an $A(\Theta)$, now we prove that every join-irreducible $A(\Theta)$ is an $A(r)$. Let $\Theta \in \Theta(K)$ and let $r'$ be an atom of $K$ such that $r' \equiv 0\ (\Theta)$. Denote by $\Theta(r')$ the least congruence relation under which $r' \equiv 0$. Obviously $\Theta = \bigvee(\Theta(r');\ r' \equiv 0\ (\Theta))$; hence if $\Theta$ is join-irreducible, it follows that $\Theta = \Theta(r')$. We prove that $p \in A(r)$ implies $p \equiv 0\ (\Theta)$. Two facts must be proved. First: $p' \equiv 0\ (\Theta)$ implies $p'' \equiv 0\ (\Theta)$, secondly: $p \prec q$, $q' \equiv 0\ (\Theta)$ imply $p' \equiv 0\ (\Theta)$. These two assertions prove the above one mentioned because by the finiteness of $P$ $p < r$ implies the existence of a chain $p = p_1 \prec p_2 \prec \ldots \prec p_n = r$ and an $n$-fold application of the two assertions implies $p' \equiv 0\ (\Theta)$. To prove the first assertion we may suppose $p' \neq p''$, and then there is an $s \in P$ such that $p \prec s$; $p' \equiv 0\ (\Theta)$ implies $p' \cup s' \equiv s'(\Theta)$ and by (1) $p'' \leqq p \cup s'$, thus $p'' = p'' \cap (p' \cup s') \equiv p'' \cap s' = 0\ (\Theta)$. Now we prove the second assertion: $q' \equiv 0\ (\Theta)$ implies $p' \cup q' \equiv p'\ (\Theta)$ and $p'' \cup q' \equiv p''\ (\Theta)$; but $p' < p'' \cup q'$,

$p'' < p' \cup q'$ by (1), thus $p' \cup q' = p'' \cup q'$. Taking the meet of the two congruences we get $p'' \cup q' \equiv 0\ (\Theta)$, and meeting by $p''$ we conclude $p'' \equiv 0\ (\Theta)$, as desired.

To sum up: there is a one-to-one order preserving correspondence between the join-irreducible congruences of $K$ and the $A(r)$ and between the $A(r)$ and $r$; hence the partially ordered set of join-irreducible congruences is isomorphic to $P$, finishing the proof of (ii) of Theorem 1.

In [7] we have proved that in a section complemented lattice the congruences are permutable; this establishes (v) of Theorem 1.

Instead of part (iv) of Theorem 1 we prove the assertion of the footnote to part (iv). Let $L = L_1 \times L_2$ and $P_1 = P \wedge L_1$, $P_2 = P \wedge L_2$. Then $P = P_1 \vee P_2$ and $x \in P_1$, $y \in P_2$ imply that $x$ and $y$ are incomparable, in symbol $P = P_1 + P_2$ (cardinal sum). This obviously implies $K(L) = K(L_1) \times K(L_2)$. The converse statement may be proved in the same way.

It remained to prove (iii). $K$ is a section complemented finite lattice, consequently, its length is less than or equal to the number of atoms. We prove that $K$ has at most $2n-1$ atoms. Indeed, if $x$ denotes the number of maximal elements of $P$, then $K$ has $x + 2(n-x) = 2n - x$ atoms. The smaller is the $x$ the greater is the number of atoms of $K$. The least value of $x$ is 1, so $K$ has at most $2n-1$ atoms. The estimation is the best possible, for if $L = 2^{n-1} + 1$, i. e. $L$ is a Boolean algebra of $n-1$ atoms, with a new unit element adjoined, then the length of $L$ is $n$ and the length of $K$ is exactly $2n-1$.

## § 3. The infinite case

Let $P$ be an arbitrary partially ordered set, and define $H$ by taking every element of $P$ in two copies $p^1, p^2$. We agree again in putting $p^1 = p^2$ if and only if $p$ is maximal in $P$. The subset $G \subseteqq H$ is closed if

$$(1') \qquad p < q, \quad p', q' \in G \quad \text{implies} \quad p'' \in G.$$

We define $K$ as the lattice of all *finite* closed subsets of $H$. $K$ is a lattice, for the closure of a finite subset of $H$ is finite again. One can prove in the very same way as in the finite case that $K$ is a section complemented lattice.

To every $r \in P$ we define the ideal $\mathfrak{A}(r)$ consisting of all $A \in K$ such that $p' \in A$ implies $p \geqq r$. The reasoning that proved in the finite case that $A(r)$ is a distributive element, proves now that $\mathfrak{A}(r)$ is a distributive ideal. Further, if $r \equiv 0\ (\Theta)$ in $K$, then $A \equiv 0\ (\Theta)$ for every $A \in \mathfrak{A}(r)$. This implies that the join-irreducible compact congruence relations are just those which are determined by the $\mathfrak{A}(r)$, further, $K$ is locally finite, thus every compact congruence relation is a finite join of join-irreducible ones. It follows that $\Theta(K)$ is isomorphic to $2^P$.

Several characterizations of lattices which occur in Theorem 2 are given in Theorem 3. Now we prove the equivalence of conditions (i)—(v) of Theorem 3.

The equivalence of (ii) and (iii) is a special case of a theorem of Büchi [3]. The equivalence of (iv) and (v) follows from a theorem of Nachbin [10]. Hence it is enough to prove the implications (i) → (ii) → (iv) → (i).

It is easy to verify that $2^P$ is isomorphic to $S(P)$, where $S(P)$ denotes the set of all $s$-ideals of $P$. An $s$-ideal $H$ of $P$ is a subset such that $x \in H$ and $y \leqq x$ imply $y \in H$. $S(P)$ is a complete lattice in which the complete join and meet coincide with the set-theoretical join and meet.

Now if $L \cong 2^P$, then $L \cong S(P)$, the latter being a complete sublattice of the complete atomic Boolean algebra of all subsets of $P$; thus (i)$\rightarrow$(ii) is proved.

If $L$ is a complete sublattice of an atomic complete Boolean algebra, then $L$ is complete and for any $p \in B$, $p$ is of finite height, thus we may take the least element $A(p)$ of $L$ which is $\geqq p$. It is routine to check that an element of $L$ is compact if and only if it is of the form $A(p)$. Further, $A(p)$ is join-irreducible if and only if $p$ is an atom, and if $p = \bigvee_{i=1}^{n} p_i$, where the $p_i$-s are atoms, then $A(p) = \bigvee_{i=1}^{n} A(p_i)$; thus (ii)$\rightarrow$(iv) is completely proved.

Finally, if $L$ satisfies (iv), then let $P$ denote the partially ordered set of join-irreducible compact elements of $L$. The proof of $L \cong S(P)$ is straight forward. Then, on using the note we made at the beginning of the proof, we see $L \cong 2^P$, finishing the proof of (iv)$\rightarrow$(i) and of Theorem 3.

Theorem 4 is nothing else but an application of Theorem 3. If $a, b \in K$, we denote by $\Theta_{ab}$ the least congruence relation under which $a \equiv b$. A congruence relation $\Theta$ is compact if and only if it may be written in the form $\Theta = \bigvee_{i=1}^{n} \Theta_{a_i b_i}$.

$\Theta(K)$ is $\bigvee$-distributive [1], and every element is the meet of completely meet-irreducible elements [2], further $\Theta(K)$ is compactly generated.

$\Theta_{ab}$ $(a \leqq b)$ is join-irreducible if and only if the interval $[a, b]$ is irreducible (the irreducibility of an interval being defined before Theorem 4).

Thus for a $\Theta(K)$ condition (1) of Theorem 4 is the same as (iv) of Theorem 3; (2) of Theorem 4 is identical with the *dual* of (iii) of Theorem 3; (3) is the same as (i) in Theorem 3; and (4) is equivalent to (ii) of Theorem 3. But condition (ii) of Theorem 3 is self-dual, hence not only the conditions of Theorem 3 are equivalent, but they are also equivalent to their duals. We infer that the conditions of Theorem 4 are equivalent.

$2^P$ is a Boolean algebra if and only if $P$ is unordered. Thus $\Theta(L)$ is a Boolean algebra if $\Theta$ and $\Phi$ are compact join-irreducible congruences, then neither $\Theta > \Phi$ nor $\Theta < \Phi$ does hold. But a join-irreducible $\Theta = \Theta_{ab}$ has this property if and only if it is minimal in the sense defined before Theorem 4. Thus Corollary 1 is proved.

If $K$ is distributive, $a < b$, $a, b \in L$, then $\Theta_{ab} = \Theta_{ac} \cup \Theta_{bc}$ with every $a < c < b$ and $\Theta_{ac}$, $\Theta_{bc} < \Theta_{ab}$. It follows that the following three conditions are equivalent: 1. $\Theta_{ab}$ is irreducible; 2. $\Theta_{ab}$ is minimal; 3. $[a, b]$ is a prime interval, i. e. no $c$ exists with $a < c < b$. Now Corollary 2 is trivial.

Corollary 3 does not call for proof.

MATHEMATICAL INSTITUTE,
HUNGARIAN ACADEMY OF SCIENCES,
BUDAPEST

*(Received January 16, 1961)*

## Bibliography


[1] G. Birkhoff, *Lattice theory* (New York, 1948).
[2] G. Birkhoff and O. Frink, Representations of lattice by sets, *Trans. Amer. Math. Soc.*, **64** (1948), pp. 299—316.
[3] I. R. Büchi, Representation of complete lattices by sets, *Porth. Math.*, **11** (1952), pp. 151—163.
[4] P. Crawley, Lattices whose congruences form a Boolean algebra, *Pacific J. of Math.*, **10** (1960), pp. 787—798.
[5] G. Grätzer and E. T. Schmidt, Two notes on lattice congruences, *Annales Univ. Sci. Budapest*, Sectio Math., **1** (1958), pp. 83—87.
[6] G. Grätzer and E. T. Schmidt, Ideals and congruence relations in lattices, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 137—175.
[7] G. Grätzer and E. T. Schmidt, Standard ideals, *Acta Math. Acad. Sci. Hung.*, **12** (1961), pp. 17—86.
[8] J. Hashimoto, Ideal theory for lattices, *Math. Japonicae*, **2** (1952), pp. 149—186.
[9] J. Hashimoto, Direct, subdirect decompositions and congruence relations, *Osaka Math. Journal*, **9** (1957), pp. 87—117.
[10] L. Nachbin, On a characterization of the lattice of all ideals of a Boolean ring, *Fund. Math.*, **36** (1949), pp. 137—142.
[11] O. Ore, Theory of equivalence relations, *Duke Math. Journal*, **9** (1942), pp. 573—627.