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MATHEMATICS

ON A PROBLEM OF L. FUCHS CONCERNING UNIVERSAL SUBGROUPS AND UNIVERSAL HOMOMORPHIC IMAGES OF ABELIAN GROUPS

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In this note our aim is to prove the following

Theorem. A necessary and sufficient condition for an abelian group G to contain a universal subgroup [1]) Z such that G/Z is a universal homomorphic image [2]) is

(a) for p-groups: $\mathfrak{r}(p^iG)$ is O or infinite, for every integer i;
(b) for torsion groups: every p-component of G fulfils (a);
(c) for groups with torsion free rank $\mathfrak{r}(>0)$: $\mathfrak{r}$ is infinite.

This theorem solves completely a problem of L. FUCHS ([1], p. 352, Problem 85).

For convenience sake we call a universal subgroup Z *perfect* if G/Z is a universal homomorphic image.

Remark 1. Condition (a) means the following: G is either bounded and $p^{k-1}G$ is infinite, where k is the smallest integer with $p^kG=0$, or G is unbounded and the final rank of G ($\text{fin}\ \mathfrak{r}(G)=\min\limits_i \mathfrak{r}(p^iG)$) is infinite.

Remark 2. Comparing our theorem with a theorem of L. FUCHS [3]) we find that the existence of universal subgroups implies in almost all cases the existence of a perfect universal subgroup Z. The exceptions are: (a_1) G is a bounded p-group such that $p^{k-1}G$ is finite but not 0; (b_1) G is a torsion group every p-component of which fulfills (a) or (a_1) and at least one p-component fulfills (a_1); (c_1) the torsion free rank of G is a natural integer $\mathfrak{r}$ and $G=T+\sum\limits_{\mathfrak{r}} C(\infty)$, where T is a torsion group satisfying (b) or (b_1).

For the notions and notations we refer to [1].

We need the trivial

Lemma. If Z_1 is a universal subgroup of G and U_1 a universal homomorphic image and if we have for a subgroup Z of G

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[1]) A subgroup Z of G is a universal subgroup if every subgroup of G is isomorphic to a homomorphic image of Z (see [1], p. 341, or [2]).

[2]) A homomorphic image U of G is called universal if every homomorphic image of G is isomorphic to some subgroup of U (see [1] p. 336, or [2]).

[3]) See [1], p. 343, or [2].

1°. Z_1 is a homomorphic image of Z;

2°. U_1 is isomorphic to a subgroup of G/Z,

then Z is a universal subgroup and G/Z is a universal homomorphic image of G.

Proof of the Theorem.

Case (a). *Necessity.* From the theorem of Fuchs, mentioned in remark 2, we know that fin $\mathfrak{r}(G)=0$ or infinite. Hence we need only consider the case that $\mathfrak{r}(p^kG)=0$, $0\neq\mathfrak{r}(p^{k-1}G)<\infty$. Then $Z\subseteq G$, $Z\sim G$ imply $\mathfrak{r}(p^{k-1}Z)=\mathfrak{r}(p^{k-1}G)$, thus $p^{k-1}Z=p^{k-1}G$ and it is impossible that G is isomorphic to a subgroup of G/Z.

Sufficiency. First, let G be a bounded p-group with the minimal bound p^k. Then $G=G_1+\ldots+G_k$, where $G_i=\sum C(p^i)$ and $\mathfrak{r}(G_k)=\mathfrak{r}(p^{k-1}G)$; thus the condition implies that G_k is infinite. We put $G_i=G_i'+G_i''$, where $G_i''=0$ or $G_i''\cong G_i'$ according as G_i is finite or not. Then $Z=\sum_{i=1}^{k}G_i'$ is a perfect universal subgroup, for we may choose in the Lemma $G\cong Z_1\cong U_1$, as $Z\sim G$ (in fact, $Z\cong G$); that G is isomorphic to a subgroup of G/Z is trivial.

Secondly, if G is unbounded, fin $\mathfrak{r}(G)\geqslant\aleph_0$ follows from the condition. Then we may decompose [4] $G=G_1+G_2$, where G_1 is a bounded group satisfying (a) and fin $\mathfrak{r}(G)=\mathfrak{r}(G_2)=\mathfrak{m}$. It follows that G_2 contains a subgroup F isomorphic to the free p-group [5] $F_p(\mathfrak{m})$; let B be a lower basic subgroup [6] of F. We define $Z=Z'+B$, where Z' is a perfect universal subgroup of G_1.

The case (b) is trivial.

Case (c). *Necessity.* If the torsion free rank $\mathfrak{r}(>0)$ is finite, then $Z\sim G$, $G\subseteq Z$ imply $\mathfrak{r}=\mathfrak{r}_0(G)=\mathfrak{r}_0(Z)$ and thus $\mathfrak{r}_0(G/Z)=0$, contradicting the fact that G is isomorphic to a subgroup of G/Z.

Sufficiency. Let $\mathfrak{r}$ be infinite. We may decompose [7] $G=G_1+G_2$ such that G_1 is a torsion group with bounded p-components satisfying (b), thus having a perfect universal subgroup Z' and G_2 contains subgroups $H\cong F(\mathfrak{r})$ and $H_i\cong F_{p_i}(\mathfrak{m}_i)$ where $\mathfrak{m}_i$ denotes the final rank of the p_i-component of the maximal torsion subgroup of G_2 while

$$\{H, F_1, F_2, \ldots\}\cong H+\sum_i F_i.$$

We define $Z=Z'+K+B$, where $K\subseteq H$, $H/K\cong\sum_{\mathfrak{r}}R+\sum_{\mathfrak{m}_i\leqslant\mathfrak{r}}\sum_{\mathfrak{m}_i}C(p_i^\infty)$ where the summation is for all i with $\mathfrak{m}_i<\mathfrak{t}_i=\max(\mathfrak{r},\mathfrak{m}_i)$, further B is a lower

4) This follows easily from Theorem 31.5 of [1].

5) See [1], p. 39.

6) See [1], p. 105 and Theorem 31.4.

7) Lemma 87.2 of [1].

basic subgroup of $\sum_{\mathfrak{m}_i \geqq \aleph_0} H_i$. We may choose [8] $U_1 = G_1 + \sum_{\mathfrak{r}} R + \sum_i \sum_{\mathfrak{t}_i} C(p_i^\infty)$ and $Z_1 = G_1 + F(\mathfrak{r}) + \sum_{\mathfrak{m}_i \geqq \aleph_0} F_{p_i}(\mathfrak{m}_i)$.

Now $Z \cong Z_1$ and that U_1 is isomorphic to a subgroup of G/Z is clear from $\sum_{\mathfrak{m}_i \geqq \aleph_0} H_i \cong \sum_i \sum_{\mathfrak{m}_i \geqq \aleph_0} C(p_i^\infty)$. Thus the proof of the theorem is complete.

[8]) See [1], p. 338 and p. 342.

LITERATURE

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