KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, **61**, No. 5 and Indag. Math., **20**, No. 5, 1958

MATHEMATICS

ON THE GENERALIZED BOOLEAN ALGEBRA GENERATED BY A DISTRIBUTIVE LATTICE

BY

G. GRÄTZER AND E. T. SCHMIDT

(Communicated by Prof. J. F. Koksma at the meeting of June 28, 1958)

1. *Introduction.* In this note our first aim is to prove the following theorem of J. Hashimoto [5] [1]):

Theorem 1. To any distributive lattice L there exists a generalized Boolean algebra [2]) B having the properties

(1) L is a sublattice of B;

(2) $\Theta(L)$ is [3]) isomorphic to $\Theta(B)$;

(3) if the interval $[a, b]$ of L is of finite length, then $[a, b]$ has the same length as an interval of B.

The importance of this theorem lies in the fact that it reduces the examination of $\Theta(L)$, in case L is distributive, to the special case of a generalized Boolean algebra, in which case this lattice was completely characterized by Komatu [6].

We prove this theorem in two different ways. Both proofs make no use of the Axiom of Choice, so we get two algebraic proofs of the embeddability of a distributive lattice in a Boolean algebra.

The first proof is based on a construction of Mac Neille [7]. However, as it was pointed out by Peremans [8], the proof of the correctness of Mac Neille's construction is not complete [4]).

We shall start with completing Mac Neille's proof, and then as an easy consequence we shall get Theorem 1.

Our second proof constructs B from $\Theta(L)$. We prove that $\Theta(L)$ is

---

[1]) In our paper [4] we have proved all but the above purely lattice theoretical theorems of J. Hashimoto's paper in pure lattice theoretical way. Theorem 1 is a combination of Theorems 8,3 and 8,5 of [5].

[2]) A Boolean ring is a commutative and associative ring of idempotent characteristic two ($a^2 = a$, for all a). Let B be a Boolean ring and define $a \cup b = a + b + ab$ and $a \cap b = ab$. We respect to these operations $\cup, \cap$, B becomes a relatively complemented, distributive lattice with zero element; B is called a generalized Boolean algebra. Furthermore, every generalized Boolean algebra may be constructed in such a way. We should like to point out that if we define $a \cap b = ab$ in B, then the only possible way for getting a lattice from B is the above described one.

[3]) $\Theta(L)$ denotes the lattice of all congruence relations of the lattice L (see [1]).

) Peremans writes that he has not been able to fill out the gap in the proof of Mac Neille without assuming the embeddability.

the lattice of all ideals of a generalized Boolean algebra. Our main tool is the well known theorem of KOMATU [6] (see in [1] and [2] too). We shall make use of some results from [3]. This proof leads to the following generalization of Theorem 1:

Theorem 2. The lattice of all congruence relations of a lattice L is isomorphic to the lattice of all congruence relations of a suitable generalized Boolean algebra if and only if every congruence relation of the form [5]) Θ_{ab} has a complement in $\Theta(L)$.

In [3] we have proved that a distributive lattice satisfies the hypothesis of Theorem 2, accordingly, Theorem 2 is actually a generalization of Theorem 1.

2. *The proof of Theorem* 1. Let L denote a distributive lattice with the elements $a, b, c, \ldots$. We denote also by $a, b, c, \ldots$ the generators of $\bar{B}$ which is defined as the associative ring generated by the elements $a, b, c, \ldots$ with the defining relations $2a=0$ for all $a \in L$ and $ab=c$ if $c=a \cap b$ in L. Hence $\bar{B}$ consists of 0 (the zero element of $\bar{B}$) and of all finite sums $\sum a_i$ $(a_i \in L)$.

If L may be embedded — as a sublattice — in a generalized Boolean algebra B_1, then considering the subring B_2 of B_1 generated by L, from the definition of $\bar{B}$ it follows that B_2 is a homomorphic image of $\bar{B}$. The kernel, J, of this homomorphism, contains all the elements of the form $a+b+a \cap b+a \cup b$, for in B_2 the identity $a+b+a \cap b+a \cup b=0$ is satisfied (this identifies the join operation of L with that of B_2). The subring I_L of $\bar{B}$ generated by the elements of the type $a+b+a \cap b+a \cup b$ is an ideal (owing to the identity

$$c(a+b+a \cup b+a \cap b)=ca+cb+ca \cup cb+ca \cap cb,$$

which is a consequence of the distributivity of L). Obviously, J and therefore I_L does not contain elements of the form a ($a \in L$ and a is not equal to the zero o of L if it exists) or $a+b$ $(a, b \in L,\ a \neq b)$ for L is a sublattice of B_2 and so $a=o$ or $a=b$ in B_2 is impossible. On the other hand, if I_L does not contain elements of the above type, then — identifying the elements of L with the generators of $\bar{B}/I_L$ — L becomes a sublattice of $\bar{B}/I_L$. Hence we get

L may be imbedded in a generalized Boolean algebra if and only if I_L does not contain elements of the form a $(a \neq o)$ and $a+b$ $(a \neq b)$.

Now let us suppose that in case of a distributive lattice L the ideal I_L contains an element x of the type $a(\neq o)$ or $a+b$ $(a \neq b)$. Then there exists a finite number of elements a_i and b_i such that

$$x=\sum_{i=1}^{n}(a_i+b_i+a_i \cup b_i+a_i \cap b_i).$$

[5]) Θ_{ab} denotes the congruence relation induced by $a \equiv b$, in other words, the minimal congruence relation Θ with $a \equiv b(\Theta)$.

Let D be the sublattice of L generated by these a_i and b_i. By the construction of D and from the italicized assertion it follows that D can neither be embedded in a generalized Boolean algebra (for $x \in I_D$). But D is finite so we have got a contradiction [6]).

Thus we have proved the embeddability of distributive lattices in generalized Boolean algebras.

Let B denote the generalized Boolean algebra $\bar{B}/I_L$, if L has no zero element; otherwise let B be the homomorphic image of $\bar{B}/L_L$ obtained by adjoining the new relation $o=0$. We prove that B fulfils the requirements of Theorem 1.

Property (1) was already proved in the previous paragraphs.

Property (3) may be proved directly by a little computation, but we can avoid it by remarking that if (3) failed to be true in the distributive lattice L, then it would not be valid even in some finite sublattice of L, a contradiction [6]).

In proving (2) we shall make use of the following lemma of MAC NEILLE [7]:

Lemma 1. Every element x of B may be written in a standard form $x = \sum_{i=1}^{n} a_i$, where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ $(a_i \in L)$.

Proof.[7]) The case $n=1$ is trivial. We use induction on n, that is, we suppose that $a_2 \leqslant \ldots \leqslant a_n$. By a repeated use of the identity

$$a+b+a \cup b+a \cap b=0,$$

we get

$$x = a_1 \cap a_2 + (a_1 \cup a_2) \cap a_3 + (a_1 \cup a_2 \cup a_3) \cap a_4 + \ldots + (a_1 \cup a_2 \cup \ldots \cup a_n),$$

completing the proof.

We use Lemma 1 in order to prove

Lemma 2. Let I, J be two ideals of B such that $I \supset J$. There exists an equality of the form $a=0$ or $a=b$ $(a, b \in L)$, which holds in B/I but not in B/J.

Proof. Let [8]) $x \in I \setminus J$ and let $x = \sum_{i=1}^{n} a_i$ $(a_1 \leqslant \ldots \leqslant a_n)$ be of standard form. We may assume that $a_1 \notin J$ and $a_1 + a_2 \notin J$. Indeed, there exists a least a_j with $a_j \notin J$, for $a_n \in J$ implies $x \in J$, a contradiction. If $a_1 + a_2 \in J$, then we consider $x + a_1 + a_2$ and proceed thus until we get an element of the required form or a contradiction to $x \in I \setminus J$.

---

[6]) We have supposed that the reader is familiar with Theorem 1 in case of a finite distributive lattice. Then B may be constructed as the Boolean algebra of all subsets of the set of the meet-irreducible elements of L. The embedding is $a \to \{x; x \text{ is meet-irreducible}, x \geqslant a\}$. Conditions (1)–(3) may be easily verified (naturally without transfinite methods), but we shall refer only to (1) and (3).

[7]) This proof is that of [7].

[8]) $\setminus$ denotes the set-theoretical difference.

If n is odd, then in B/I a new equality is $a_1=0$, for $0=x=xa_1=na_1=a_1$. In case n is even, then $a_1=a_2$ is a required one which is valid for $0=x=xa_2=a_1+a_2$. These identities fail to be true in B/J for $a_1 \notin J$ and $a_1+a_2 \notin J$ were supposed.

Obviously, a congruence relation Θ in L induces a congruence relation $\bar{\Theta}$ in B, if we identify the generators a, b of B if and only if $a \equiv b\ (\Theta)$. The relations Θ and $\bar{\Theta}$ coincide on L. Thus different congruence relations of L may be extended to different congruence relations of B. In order to complete the proof of (2) it remains only to show that different congruence relations of B are different on L. But this is an immediate consequence of Lemma 2. Thus the proof of Theorem 1 is completed.

Let us note that the special case $J=(0)$ of Lemma 2 has been proved by MAC NEILLE [7]. This special case leads to the following important assertion:

Corollary. (Theorem of Mac Neille). B is the smallest generalized Boolean algebra in which L may be embedded, that is, no sublattice or homomorphic image of B contains L as a sublattice.

3. *The proof of Theorem* 2. First we recall some definitions.

Let H be a complete lattice. The subset $\{x_\alpha\}$ of H is called a directed set if given x_α and x_β some x_γ satisfies $x_\alpha \leqslant x_\gamma$ and $x_\beta \leqslant x_\gamma$. It follows readily that every finite subset of $\{x_\alpha\}$ has upper bounds within $\{x_\alpha\}$. If $\{x_\alpha\}$ is a directed set and $\bigcup\limits_\alpha x_\alpha = x$, then we write $x_\alpha \uparrow x$. If, for a fixed x, $x_\alpha \uparrow x$ implies that some x_α equals x, then we say that x is $\uparrow$-inaccessible [9]) (or x is inaccessible from below).

If $\{x_\alpha\}$ is a subset of H, then the subset $[x_\alpha]$ is called the natural directed extension of $\{x_\alpha\}$, if it consists of all finite joins of the x_α. Naturally, $[x_\alpha]$ is a directed set.

First of all we prove the following

Lemma 3. Let L be a lattice (or an arbitrary algebra with finitary operations [10]). The element Θ of $\Theta(L)$ is $\uparrow$-inaccessible if and only if it is of the form $\Theta = \bigvee\limits_{i=1}^{n} \Theta_{a_i b_i}$.

Proof. Let $\Theta = \bigvee\limits_{i=1}^{n} \Theta_{a_i b_i}$ and $\Theta_\alpha \uparrow \Theta$. Since $a_i \equiv b_i\ (\bigvee\limits_\alpha \Theta_\alpha)$, for some finite subset Θ_j^i of the Θ_α we have the relation $a_i \equiv b_i\ (\bigvee\limits_j \Theta_j^i)$. Let $\Phi \in \{\Theta_\alpha\}$ be an upper bound for the Θ_j^i $(i, j = 1, 2, \ldots)$. Then $a_i \equiv b_i\ (\Phi)$ $(i=1, 2, \ldots, n)$. Consequently, $\Phi \geqslant \Theta$. On the other hand $\Phi \in \{\Theta_\alpha\}$ and so $\Phi \leqslant \Theta$, it follows that $\Phi = \Theta$.

Now let Θ be $\uparrow$-inaccessible. Obviously, $\Theta = \bigvee\limits_{a \equiv b(\Theta)} \Theta_{ab}$, hence the natural directed extension of these Θ_{ab} accesses Θ. Thus $\Theta = \bigvee\limits_{i=1}^{n} \Theta_{a_i b_i}$, and the proof of Lemma 3 is completed.

9) See [2].

10) In the sense of [1].

Now we are able to prove Theorem 2.

Let L be a (not necessarily distributive) lattice and let us suppose that there exists a generalized Boolean algebra B with $\Theta(L) \cong \Theta(B)$. As it is well known, $\Theta(B)$ is isomorphic to the lattice $\mathfrak{B}$ of all ideals of B. By Lemma 3, the $\uparrow$-inaccessible elements of $\Theta(L)$ are of the form $\bigvee_{i=1}^{n} \Theta_{a_i b_i}$, and it is well known that the $\uparrow$-inaccessible elements of $\mathfrak{B}$ are just the principal ideals of B [11]). Hence under any isomorphism $\Theta(L) \cong \mathfrak{B}$ the elements of the form $\bigvee_{i=1}^{n} \Theta_{a_i b_i}$ correspond to the principal ideals of B, for under isomorphism the $\uparrow$-inaccessibility is preserved. Consequently, if we prove that in $\mathfrak{B}$ any principal ideal of B has a complement, then we know the same for the elements of $\Theta(L)$ of the form $\bigvee_{i=1}^{n} \Theta_{a_i b_i}$, hence, in particular, for all Θ_{ab}.

Let $(a]$ be a principal ideal of the generalized Boolean algebra B. Define K as the set of all x satisfying $a \cap x = 0$. From the distributivity of B we get that K is an ideal, while $(a] \cap K = 0$ is obvious. Let u be arbitrary in B and u_a the relative complement of $a \cap u$ in the interval $[0, u]$. Because of $a \cap u_a = 0$ it follows $u_a \in K$. Furthermore $u \cap a \in (a]$, hence $u = u_a \cup (u \cap a) \in K \cup (a]$. Thus, K is the complement of $(a]$ in $\mathfrak{B}$.

Now, we suppose that in $\Theta(L)$ every Θ_{ab} has a complement Θ'_{ab}. We prove that both $\Theta_{ab} \cap \Theta_{cd}$ and $\Theta'_{ab} \cap \Theta_{cd}$ are $\uparrow$-inaccessible for all $a, b, c, d \in L$. We may suppose $a < b$, $c < d$. There is a chain

$$c = x_0 < x_1 < \ldots < x_n = d$$

such that for every i either $x_{i-1} \equiv x_i(\Theta_{ab})$ or $x_i \equiv x_{i-1}(\Theta'_{ab})$ (see [3]). Let us denote by p_i the intervals of the first type and by q_j those of the second type. Obviously,

$$\bigvee_{i=1}^{k} \Theta_{p_i} \cup \bigvee_{j=1}^{l} \Theta_{q_j} = \Theta_{cd} \text{ and [12]) } \Theta_{ab} \cap \Theta_{q_j} = \omega$$

(for all j). We have

$$\Theta_{ab} \cap \Theta_{cd} = \Theta_{ab} \cap (\bigvee_i \Theta_{p_i} \cup \bigvee_j \Theta_{q_j}) = (\Theta_{ab} \cap \bigvee_i \Theta_{p_i}) \cup \bigvee_j (\Theta_{ab} \cap \Theta_{q_j}) = \bigvee_{i=1}^{k} \Theta_{p_i}$$

and in the same way we get

$$\Theta'_{ab} \cap \Theta_{cd} = \bigvee_{j=1}^{l} \Theta_{q_j},$$

and our assertion follows by Lemma 3.

We prove that the $\uparrow$-inaccessible elements of $\Theta(L)$ form a relatively complemented sublattice with zero element. From the identity

$$\bigvee_i \Theta_{a_i b_i} \cap \bigvee_j \Theta_{c_j d_j} = \bigvee_{i,j} (\Theta_{a_i b_i} \cap \Theta_{c_j d_j})$$

---

11) See in [6] or also in [1] and [2].

12) ω denotes the zero of $\Theta(L)$.

it follows the property of being a sublattice. $\omega = \Theta_{a,a}$ is an element of this sublattice. The relative complementedness may be proved easily, for let

$$\bigvee_{i=1}^{n} \Theta_{a_i, b_i} \leqq \bigvee_{j=1}^{m} \Theta_{c_j d_j},$$

then the relative complement of $\bigvee \Theta_{a_i b_i}$ in the interval $[\omega, \bigvee \Theta_{c_j d_j}]$ is

$$\bigwedge_{i=1}^{n} \Theta'_{a_i b_i} \cap \bigvee_{j=1}^{m} \Theta_{c_j d_j},$$

the $\uparrow$-inaccessibility of which may be proved from the result of the previous paragraph by an easy induction on $n+m$.

Now, we turn to the theorem of KOMATU [6] in order to prove that the generalized Boolean algebra B of the $\uparrow$-inaccessible elements of $\Theta(L)$ satisfies the condition $\Theta(B) \cong \Theta(L)$.

Komatu's theorem (see [6], or in [1] and [2], too) asserts: Let H be a lattice. H is the lattice of all ideals of a suitable lattice if and only if the following conditions are satisfied: (i) H is complete; (ii) every element of H is join of $\uparrow$-inaccessible elements; (iii) $x_\alpha \uparrow x$ implies $x_\alpha \cap y \uparrow x \cap y$; (iv) the $\uparrow$-inaccessible elements of H form a sublattice L. Furthermore, if (i)–(iv) are satisfied then H is the lattice of all ideals of L.

Conditions (i)–(iii) hold in $\Theta(L)$ (this was proved in [2], but in this special case this may be readily verified owing to Lemma 3, to the distributivity of $\Theta(L)$ and to Birkhoff's theorem – see [2], p. 23 – which assures (i)). Hence it follows that $\Theta(L)$ is isomorphic to the lattice of all ideals of B, completing the proof of Theorem 2.

As immediate consequences of Theorem 2 we have

Corollary 1. Let L be a lattice. There exists a Boolean algebra B with $\Theta(L) \cong \Theta(B)$ if and only if every congruence relation of the form Θ_{ab} has a complement in $\Theta(L)$ and for some $u, v \in L$, Θ_{uv} is the greatest element of $\Theta(L)$.

Corollary 2. Let L be a distributive lattice. There exists a Boolean algebra B with $\Theta(L) \cong \Theta(B)$ if and only if L has a least and a greatest element.

Corollary 1 is obvious. Corollary 2 is a consequence of Corollary 1, for in a distributive lattice all Θ_{ab} in $\Theta(L)$ are complemented (see [3]) and if Θ_{uv} $(u<v,\, u,\, v \in L)$ is the greatest element of $\Theta(L)$ and e.g. $x<u$, then $\Theta_{xu} \cap \Theta_{uv} = \omega$ (see [3]), a contradiction.

Let us remark that a distributive lattice L with the zero element o (if L has no zero, we adjoin it to L) may be easily embedded in the generalized Boolean algebra B of the $\uparrow$-inaccessible elements of $\Theta(L)$. Indeed, the correspondence $a \to \Theta_{oa}$ is an isomorphism and carries L into a subset of B which is a sublattice (these assertions follow from the following identities of [3]: $\Theta_{oa} \cup \Theta_{ob} = \Theta_{oa \cup b}$; $\Theta_{oa} \cap \Theta_{ob} = \Theta_{oa \cap b}$).

Finally, we mention the following question:

What is a necessary and sufficient condition for $\Theta(L)$ to be isomorphic to the lattice of all ideals of a suitable lattice? [13]) Is the following condition suitable: all congruence relations of the form Θ_{ab} are separable (in the sense of [3])? Since every congruence relation having a complement is separable, this condition is a natural generalization of that of Theorem 2.

[13]) Naturally, the condition is equivalent—owing to Komatu's theorem—to the following trivial one: $\Theta_{ab} \cap \Theta_{cd}$ may be written in the form $\bigvee_{i=1}^{n} \Theta_{a_i, b_i}$.

BIBLIOGRAPHY

1. Birkhoff, G., Lattice theory, 2nd ed., New York (1948).
2. ——— and O. Frink, Representation of lattices by sets, Trans. Amer. Math. Soc., **64**, 299–316 (1948).
3. Grätzer, G. and E. T. Schmidt, Ideals and congruence relations in lattices, Acta Math. Acad. Sci. Hung., **9**, 137–175 (1958).
4. ——— and ———, On ideal theory for lattices, Acta Sci. Math. Szeged, **19**, 82–92 (1958).
5. Hashimoto, J., Ideal theory for lattices, Math. Japon., **2**, 149–186 (1952).
6. Komatu, A., On a characterization of join-homomorphic transformation lattices, Proc. Imp. Acad. Tokyo, **19**, 119–124 (1943).
7. Mac Neille, H. M., Lattices and Boolean rings, Bull. Amer. Math. Soc., **45**, 73–81 (1945).
8. Peremans, W., Embedding of a distributive lattice into a Boolean algebra, Indag. Math., **19**, 73–81 (1957).