# Finite distributive lattices are congruence lattices of almost-geometric lattices

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Abstract. A semimodular lattice $L$ of finite length will be called an *almost-geometric lattice*, if the order $J(L)$ of its nonzero join-irreducible elements is a cardinal sum of at most two-element chains. We prove that each finite distributive lattice is isomorphic to the lattice of congruences of a finite almost-geometric lattice.

## 1. Introduction

We say that an order $P = (P, \leq)$ is a cardinal sum of at most two-element chains, if for every $a \in P$, both of $\downarrow a = \{x \in P : x \leq a\}$ and $\uparrow a = \{x \in P : x \geq a\}$ are at most two-element. By an *almost-geometric lattice* we mean an (upper) semimodular lattice $L$ of finite length such that $J(L)$, the set of non-zero join-irreducible elements of $L$, is a cardinal sum of at most two-element chains. Notice that in an almost-geometric lattice, each join-irreducible element is of height at most two. The converse is not true: if we obtain $L$ by adding a new 0 to the four-element Boolean lattice, then every $a \in J(L)$ is of height at most two but $L$ is *not* an almost-geometric lattice.

Geometric lattices of finite length are almost-geometric and simple. Hence we cannot drop "almost" from our main result, the following representation theorem.

**Theorem 1.** *Every finite distributive lattice $D$ is isomorphic to the congruence lattice of a finite almost-geometric lattice $G$.*

**Notation.** We use the notation of Grätzer [4]. The Glossary of Notation of [4] is available as a pdf file at

`http://mirror.ctan.org/info/examples/Math_into_LaTeX-4/notation.pdf`

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## 2. Proofs and auxiliary statements

Firstly, we recall some notions and three lemmas. The idea of the proof will be outlined right after the "proof" of Lemma 4.

In this section, all lattices are assumed to be finite. Let $L$ be a lattice. The set of atoms of $L$ will be denoted by $A(L)$. For $x, y \in L$ with $x \parallel y$, the four-element sublattice $H = \{x \wedge y, x, y, x \vee y\}$ is called a *square* of $L$. We call $H$ a *lower covering square*, if $x \wedge y \prec x$ and $x \wedge y \prec y$. *Upper covering squares* are defined dually. By a *covering square* we mean a square that is both lower covering and upper covering. It is well-known that a finite lattice $L$ is semimodular iff it has the following property:

$$\text{every lower covering square of } L \text{ is upper covering.} \tag{1}$$

Following [10] and [11], by a *chopped lattice* we mean a partial algebra $C = (C, \wedge, \vee)$ such that $\wedge$ is a (meet-)semilattice operation and $\vee$ is a partial operation such that $a \vee b$ is defined iff $\{a, b\}$ has a least upper bound, and $a \vee b$ equals this least upper bound, provided that it exists. By an *ideal* of $C$ we mean an order-ideal closed with respect to existing joins. The ideals of $C$ form a lattice denoted by $\mathrm{Id}\, C$. Via the canonical identification of $x \in C$ with $\downarrow x \in \mathrm{Id}\, C$, we usually assume that $C \subseteq \mathrm{Id}\, C$; this way the (partial) operations of $C$ are the restrictions of the operations of $\mathrm{Id}\, C$.

Given a finite chopped lattice $C$, let $\mathrm{Max}\, C = \{t_1, \dots, t_k\}$ denote the set of its maximal elements. Notice that $k = |\mathrm{Max}\, C| = 1$ iff $C$ is a lattice. A $k$-tuple $\vec{x} = (x_1, \dots, x_k) \in \downarrow t_1 \times \dots \times \downarrow t_k$ is called a *compatible vector*, if, for all $1 \le i < j \le k$,

$$x_i \wedge t_j = x_j \wedge t_i. \tag{2}$$

Notice that in the important particular case when $t_i \wedge t_j = a$ is an atom, (2) is equivalent to the following, more manageable condition

$$\text{either } a \le x_i \text{ and } a \le x_j, \text{ or } a \not\le x_i \text{ and } a \not\le x_j. \tag{3}$$

If $t_i \wedge t_j = 0$, then (2) holds automatically. Let $\mathrm{Cmpv}\, C$ denote the set of compatible vectors. For $\vec{x}, \vec{y} \in \mathrm{Cmpv}\, C$, let $\vec{x} \le \vec{y}$ mean that $x_i \le y_i$ for $i = 1, \dots, k$. Then $\mathrm{Cmpv}\, C = (\mathrm{Cmpv}\, C, \le)$ is a finite order.

**Lemma 2** (Lemma 4.4 and its surrounding in Grätzer [4])**.**

- $\mathrm{Cmpv}\, C$ *is a lattice, and it is isomorphic with* $\mathrm{Id}\, C$*.*
- *The meet in* $\mathrm{Cmpv}\, C$ *is defined componentwise.*
- $\downarrow t_j$*, which is a lattice, is embedded in* $\mathrm{Cmpv}\, C$ *in the following canonical way:*

$$\downarrow t_i \to \mathrm{Cmpv}\, C, \quad x \mapsto \tilde{x} := (x \wedge t_1, \dots, x \wedge t_k). \tag{4}$$

Based on this lemma, we will always replace Id with the more comfortable Cmpv in what follows. The congruences of a chopped lattice $C$ are, by definition, congruences $\Theta$ of $(C_m, \wedge)$ such that if both $x_1 \vee x_2$ and $y_1 \vee y_2$ are defined and $(x_1, y_1), (x_2, y_2) \in \Theta$, then $(x_1 \vee x_2, y_1 \vee y_2) \in \Theta$.

**Lemma 3** ([10], see also Thm. 4.6 in Grätzer [4])**.** *Let $C$ be a finite chopped lattice. Then* $\mathrm{Cmpv}\, C$ *is a congruence-preserving extension of $C$. Consequently,* $\mathrm{Con}(\mathrm{Cmpv}\, C) \cong \mathrm{Con}\, C$.

Let $Q$ be a finite lattice with two distinguished atoms $p$ and $q$. Then $Q = (Q, \wedge, \vee, p, q)$, a lattice with two constants, will be called a *basic gadget*, if

- $Q$ has exactly three congruences, $\omega_Q < \mu_Q < \iota_Q$,
- $\mathrm{con}(0, p)$, the smallest congruence collapsing 0 and $p$, is $\iota_Q = Q^2$, and
- $\mathrm{con}(0, q) = \mu_Q$.

For example, $T_0 = (T_0, \wedge, \vee, p_1', q)$, the right-hand lattice in Figure 1, is a basic gadget. (The only nontrivial congruence is indicated by dotted ovals.)

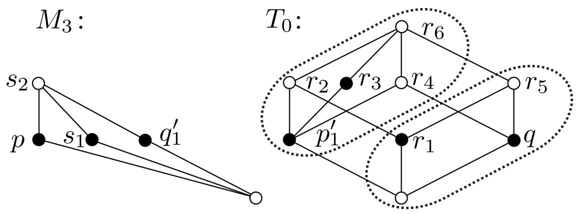


FIGURE 1. $M_3$ and $T_0$

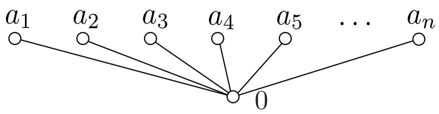


FIGURE 2. $M_0^\wedge$

Next, we consider the meet-semilattice

$$M_0^\wedge = \{0, a_1, \dots, a_n\}, \tag{5}$$

given in Figure 2. Let $G_m = \{(b_1, c_1), \dots, (b_m, c_m)\}$ be an $m$-element irreflexive and antisymmetric relation on $\{a_1, \dots, a_n\}$. That is, $b_i, c_i \in \{a_1, \dots, a_n\}$ such that $b_i \neq c_i$ for $i = 1, \dots, m$, and $i \neq j$ implies $\{b_i, c_i\} \neq \{b_j, c_j\}$. For $i = 1, \dots, m$, let $S_i$ be a finite lattice with distinct atoms $p_i, q_i$. For each $i \in \{1, \dots, m\}$, we glue $S_i$ to $M_0^\wedge$ by the identifications $0_{M_0^\wedge} = 0_{S_i}$, $b_i = p_i$ and $c_i = q_i$, but collapsing nothing else. This way we obtain a chopped lattice

$$C_m = \Big(M_0^\wedge \cup \bigcup_{i=1}^{m} S_i; b_1 = p_1, c_1 = q_1, \dots, b_m = p_m, c_m = q_m\Big). \tag{6}$$

Since this construction plays the main role in [10], we call this chopped lattice a *1960-merging* of the lattices $S_i$. Similarly, the lattice

$$\Big[M_0^\wedge \cup \bigcup_{i=1}^{m} S_i; b_1 = p_1, c_1 = q_1, \dots, b_m = p_m, c_m = q_m\Big] := \mathrm{Cmpv}\, C_m \tag{7}$$

is called a *1960-amalgam* of the lattices $S_i$.

Let $D$ be a finite distributive lattice, and choose the meet-semilattice $M_0^\wedge = \{0, a_1, \ldots, a_n\}$ in (5) such that $J(D) = \{a_1, \ldots, a_n\}$. Notice that $J(D)$ is an antichain in $M_0^\wedge = (M_0^\wedge, \leq_{M_0^\wedge})$, and it is a suborder of $(D, \leq_D)$. The covering relation of $J(D)$ will be denoted by $\prec_{J(D)}$. Let $G_m = \{(b_1, c_1), \ldots, (b_m, c_m)\}$ be $\{(x, y) \colon x, y \in J(D) \text{ and } y \prec_{J(D)} x\}$, that is, an enumeration of the relation $\succ_{J(D)}$. Fix a basic gadget $Q = (Q, \wedge, \vee, p, q)$. For $i = 1, \ldots, m$, let $S_i = Q$, $p_i = p$ and $q_i = q$. Then the 1960-amalgam of these $S_i$ makes sense; let

$$\operatorname{Con}^{-1}(D, Q) \tag{8}$$

denote this 1960-amalgam, see (7). The crucial idea is taken from [10]:

**Lemma 4** ([10])**.** *If $D$ is a finite distributive lattice and $Q$ is a basic gadget, then $\operatorname{Con}\bigl(\operatorname{Con}^{-1}(D, Q)\bigr)$ is isomorphic with $D$.*

*Instead of the proof of Lemma 4.* For convenience, we outline the main ideas of [10] and [11]; the reader may skip this part. We refer to an excellent secondary source, Grätzer [4], also.

By Lemma 3 and $\operatorname{Con}^{-1}(D, Q) = \operatorname{Cmpv} C_m$, it suffices to prove that $D \cong \operatorname{Con} C_m$, that is, $J(D) \cong J(\operatorname{Con} C_m)$, as orders. Let $\Theta \in \operatorname{Con} C_m$. It is determined by its covering pairs, that is, by $\{(x, y) \colon x \prec y \text{ and } (x, y) \in \Theta\}$. Since each covering pair of $C_m$ belongs to some $S_i$, which is (isomorphic to) a basic gadget, we easily obtain that $\Theta$ is the join of some congruences of the form $\operatorname{con}(0, p_i)$ and $\operatorname{con}(0, q_j)$. Since $0 \prec p_i$, we obtain that $\operatorname{con}(0, p_i)$ is a join-irreducible element of $\operatorname{Con} C_m$, and so is $\operatorname{con}(0, q_i)$. Since $p_i, q_j \in \{a_1, \ldots, a_n\}$, we conclude that

$$J(\operatorname{Con} C_m) = \{\operatorname{con}(0, a_1), \ldots, \operatorname{con}(0, a_n)\}.$$

It is shown in [10] that

$$a_i \leq_D a_j \iff \operatorname{con}(0, a_i) \leq \operatorname{con}(0, a_j), \tag{9}$$

whence $J(\operatorname{Con} C_m) \cong J(D)$, which implies the lemma.

Some easy details of (9) are as follows. Let $a_i \prec_{J(D)} a_j$, and denote $\operatorname{con}(0, a_j)$ by $\Theta \in \operatorname{Con} C_m$. Let $(b_k, c_k) = (a_j, a_i) \in G_m$. Then $(0, a_j) = (0, b_k) = (0, p) \in \Theta\rceil_{S_k}$. Since $\operatorname{con}(0, p) = \iota_Q$, we obtain that $\Theta\rceil_{S_k} = \iota_{S_k}$. Hence $(0, a_i) = (0, c_k) = (0, q) \in \Theta\rceil_{S_k}$. This shows that $\operatorname{con}(0, a_i) \leq \operatorname{con}(0, a_j)$. The reverse direction is much more complex: using Lemma 4.5 of Grätzer [4], the reader can easily check that $\operatorname{con}(0, a_i)$ is the congruence $\Theta \in \operatorname{Con} C_m$ determined by the property that $(0, a_k) \in \Theta$ iff $a_k \leq_D a_i$. □

*Idea of the proof* (of Theorem 1). Armed with Lemma 4, it is sufficient to find an appropriate basic gadget $Q$ such that $\operatorname{Con}^{-1}(D, Q)$ is an almost-geometric lattice. Since the 1960-amalgam (7) is a rather complicated construction, we will reach it in $m$ easier amalgamating steps. The natural assumption that $Q$ should be almost-geometric will not be sufficient in itself. Therefore, we will construct a "perfect gadget", to be defined later. Perfect gadgets will have reasonable properties preserved by the amalgamating steps.

In order to define the simplest amalgamation we need, let $L_1$ and $L_2$ be finite lattices, and let $p_i \in A(L_i)$ for $i = 1, 2$. By the identification $p_1 = p_2$ and $0_1 = 0_2$, we obtain a chopped lattice denoted by

$$(L_1 \cup L_2; p_1 = p_2).$$

This chopped lattice is called an *atomic merging* of the lattices $L_1$ and $L_2$. An example (with slightly different notation) is given in Figure 3. Let

$$[L_1 \cup L_2; p_1 = p_2] := \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2); \tag{10}$$

we call this lattice an *atomic amalgam* of the lattices $L_1$ and $L_2$. If $x \in L_i$, then $\tilde{x}$ denotes the image of $x$ under the canonical $L_i \mapsto [L_1 \cup L_2; p_1 = p_2]$ embedding, see (4). Analogous notation will apply for other amalgams. Sometimes we identify $x$ and $\tilde{x}$; this allows us to say that the atomic merging is a generating subset of the atomic amalgam.

**Lemma 5.** *Every atomic amalgam of two finite semimodular lattices is semimodular.*

*Proof.* Let $C = (L_1 \cup L_2; p_1 = p_2)$ and $L = [L_1 \cup L_2; p_1 = p_2]$. By (3), $L$ consists of vectors $\vec{x} = (x_1, x_2) \in L_1 \times L_2$ such that

$$\text{either } x_1 \geq p_1 \text{ and } x_2 \geq p_2, \text{ or } x_1 \ngeq p_1 \text{ and } x_2 \ngeq p_2\,. \tag{11}$$

Let $\vec{p} = (p_1, p_2)$. It belongs to $L$. For $\vec{x} \in L$, $\vec{x} \ngeq \vec{p}$ is equivalent to the conjunction of $x_1 \ngeq p_1$ and $x_2 \ngeq p_2$; however, this is not necessarily true for $\vec{x} \in L_1 \times L_2$.

Let $\vee_d$ and $\vee_L$ denote the join taken in the direct product $L_1 \times L_2$ and the join taken in $L$, respectively. Similarly, the covering relation in $L_1 \times L_2$ and that in $L$ will be denoted by $\prec_d$ and $\prec_L$, respectively. (Analogous notation will be used in similar environment later.) Clearly, for $\vec{u}, \vec{s} \in L$,

$$\vec{u} \prec_L \vec{s} \text{ iff either } \vec{u} \prec_d \vec{s}, \text{ or } \vec{p} \nleq \vec{u} \text{ and } \vec{s} = \vec{u} \vee_d \vec{p} = \vec{u} \vee_L \vec{p}. \tag{12}$$

Consider an arbitrary lower covering square

$$H = \{\vec{x} = \vec{y} \wedge \vec{z},\ \vec{y},\ \vec{z},\ \vec{v} = \vec{y} \vee_L \vec{z}\} \tag{13}$$

in $L$; we have to show that it is upper covering. We will use the well-known trivial fact that $L_1 \times L_2$ is semimodular, see [1] for a bit stronger result. Let

$$\vec{w} := \vec{y} \vee_d \vec{z}\,. \tag{14}$$

Notice that

$$\vec{v} \text{ is the smallest element of } L \text{ such that } \vec{w} \leq \vec{v}. \tag{15}$$

If $\vec{y} \geq \vec{p}$ and $\vec{z} \geq \vec{p}$, then $\vec{x} \geq \vec{p}$ and $H$ is a lower covering square of $L_1 \times L_2$. Hence it is upper covering in $L_1 \times L_2$, whence it is upper covering in $L$. So, we will assume that, say

$$\vec{z} \ngeq \vec{p}, \text{ whence } \vec{x} \ngeq \vec{p}.$$

*Case* 1: $\vec{y} \ngeq \vec{p}$. Then (12) yields that $\vec{x} \prec_d \vec{y}$ and $\vec{x} \prec_d \vec{z}$. Hence, up to $y$-$z$ an 1-2 symmetry, $\vec{y} = (y_1, x_2)$ where $x_1 \prec y_1$ in $L_1$.

If $z_1 = x_1$, then $\vec{z} = (x_1, z_2)$ with $x_2 \prec z_2$ in $L_2$. Therefore, $\vec{w} = \vec{y} \vee_d \vec{z} = (y_1, z_2)$. This belongs to $L$, since $y_1 \not\geq p_1$ and $z_2 \not\geq p_2$. Hence $\vec{v} = \vec{w}$ and $H$ is a covering square in $L_1 \times L_2$, so it is a covering square in $L$, indeed.

Otherwise, if $z_1 \neq x_1$, then $\vec{z} = (z_1, x_2)$ with $x_1 \prec z_1 \neq y_1$ in $L_1$. If $\vec{w} = (y_1 \vee z_1, x_2) \in L$, then $\vec{v} = \vec{w}$ and the previous argument yields that $H$ is a covering square in $L$. Assume that $\vec{w} \notin L$, that is, $p_1 \leq y_1 \vee z_1$ and $p_2 \not\leq x_2$. By (15), we see that $\vec{v} = (y_1 \vee z_1, x_2 \vee p_2)$. Since $y_1 < y_1 \vee p_1 \leq y_1 \vee z_1 = v_1$ and, by the semimodularity of $L_1$, we have that $y_1 \prec v_1$, we conclude that $y_1 \vee p_1 = v_1$. Hence $\vec{v} = \vec{y} \vee_d \vec{p}$ and (12) implies $\vec{y} \prec_L \vec{v}$. By $\vec{y}$-$\vec{z}$ symmetry, we conclude that $\vec{z} \prec_L \vec{v}$, whence $H$ is a covering square in $L$.

*Case* 2: $\vec{y} \geq \vec{p}$. We obtain from (12) that $\vec{y} = \vec{x} \vee_L \vec{p} = \vec{x} \vee_d \vec{p}$. Then $\vec{w} = \vec{y} \vee_d \vec{z} \geq \vec{p}$ yields that $\vec{w} = \vec{v}$, whence $\vec{v} = \vec{y} \vee_d \vec{z} = \vec{p} \vee_d \vec{x} \vee_d \vec{z} = \vec{p} \vee_d \vec{z}$ implies $\vec{z} \prec_L \vec{v}$ by (12). Further, since $\vec{x} \prec_d \vec{z}$ by (12), the semimodularity of $L_1 \times L_2$ implies that $\vec{y} \prec_d \vec{w} = \vec{v}$, whence $\vec{y} \prec_L \vec{v}$. This shows that $H$ is an upper covering square in $L$. □

Let $L$ be a semimodular lattice of finite length, and let $p, q \in A(L)$. We say that $(p, x, q) \in L^3$ is *perspective triplet*, if $p \wedge x = q \wedge x = 0$ and $p \vee x = q \vee x$. (This terminology is explained by Lemma IV.3.7 in Grätzer [3].) We say that $p$ and $q$ are *non-perspective atoms*, in notation $p \not\sim q$, if $p \neq q$ and, for all $x \in L$, the triplet $(p, x, q)$ is not perspective. Clearly,

$$p \not\sim q \text{ implies that } \downarrow(p \vee q) = \{0, p, q, p \vee q\}; \tag{16}$$

this is a particular case (namely, the case $x = 0$) of the second part of the following easy lemma.

**Lemma 6.** *Let $p$ and $q$ be distinct atoms of a finite semimodular lattice $L$, and let $x \in L$. Then*

- *If $p \not\leq x$, $q \not\leq x$ and $q \leq p \vee x$, then $(p, x, q)$ is a perspective triplet.*
- *If $p \not\sim q$, $p \not\leq x \vee q$ and $q \not\leq x \vee p$, then the interval $[x, x \vee p \vee q]$ equals $\{x, x \vee p, x \vee q, x \vee p \vee q\}$, which consists of four distinct elements.*

*Proof.* Since $x \prec x \vee p$ by semimodularity and $x < x \vee q \leq x \vee p$, the first statement is evident.

To prove the second statement, let $y = x \vee p \vee q$, and notice that $x \prec x \vee p \prec y$. Assume, by way of contradiction, that $x < t < y$ but $t \notin \{x \vee p, x \vee q\}$. Then, by the well known Jordan-Hölder chain condition, $x \prec t \prec y$. Notice that $p \leq t$ is impossible, since otherwise $x \vee p \leq t \prec y$ and $x \vee p \prec y$ would contradict $x \vee p \neq t$. Hence $t \wedge p = 0$, and $t \prec t \vee p \leq y$ yields $t \vee p = y$. The same argument works for $q$ instead of $p$, so we get that $(p, t, q)$ is a perspective triplet, contradicting $p \not\sim q$. □

A basic gadget $Q = (Q, \wedge, \vee, p, q)$ is called a *perfect gadget*, if

$$p \not\sim q, \tag{17}$$

$$(Q, \wedge, \vee) \text{ is a finite almost-geometric lattice, and} \tag{18}$$

$$J(Q) \cap \uparrow p = \{p\} \text{ and } J(Q) \cap \uparrow q = \{q\}. \tag{19}$$

In order to construct a perfect gadget, we need the following two lemmas.

**Lemma 7.** *Let $L = [L_1 \cup L_2; p_1 = p_2] = \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2)$ be an atomic amalgam of two finite semimodular lattices $L_1$ and $L_2$.*

- *If $x \in A(L_1)$ and $y \in A(L_2)$, then $\tilde{x}, \tilde{y} \in A(L)$.*
- *If $a, b$ are non-perspective atoms in $L_i$, then $\tilde{a} \not\sim \tilde{b}$.*
- *If $a \in A(L_1)$, $b \in A(L_2)$, and either $a \not\sim p_1$ or $b \not\sim p_2$, then $\tilde{a} \not\sim \tilde{b}$ in $L$.*

*Proof.* The first part of the lemma is evident. Hence, in the rest of the proof, we know that $\tilde{a}, \tilde{b} \in A(L)$. By way of contradiction, we assume that $T = (\tilde{a}, \vec{x}, \tilde{b}) \in L^3$ is a perspective triplet. Let $\vec{y} = \tilde{a} \vee_L \vec{x} = \tilde{b} \vee_L \vec{x}$. Lemma 5 yields that $\vec{x} \prec_L \vec{y}$.

To prove the second part, we can assume that $i = 1$. The perspectivity of $T$ gives that $a \not\leq x_1$ and $b \not\leq x_1$.

Firstly, assume that $p_1 \in \{a, b\}$. Say, $b = p_1$, so $\tilde{b} = (p_1, p_2) = \vec{p}$ and $\tilde{a} = (a, 0)$. Then $\tilde{a} \leq \vec{y} = \vec{x} \vee_L \tilde{b} = \vec{x} \vee_L \vec{p} = \vec{x} \vee_d \vec{p}$ yields that $a \leq x_1 \vee p_1$, whence $(a, x_1, p_1) \in L_1^3$ is a perspective triplet by Lemma 6, a contradiction.

Secondly, we consider the case $p_1 \notin \{a, b\}$. Then $\tilde{a} = (a, 0)$ and $\tilde{b} = (b, 0)$. Since $\vec{x} \prec_L \vec{y}$, we can distinguish two cases according to (12).

*Case 1: $\vec{x} \prec_d \vec{y}$.* Then $\vec{x} < \tilde{a} \vee_d \vec{x} \leq \tilde{a} \vee_L \vec{x} = \vec{y}$ gives that $(a \vee x_1, x_2) = \tilde{a} \vee_d \vec{x} = \vec{y}$. Similarly, $(b \vee x_1, x_2) = \vec{y}$. Hence $(a, x_1, b)$ is a perspective triplet in $L_1$, a contradiction.

*Case 2: $\vec{p} \not\leq \vec{x}$ and $\vec{x} \vee_d \vec{p} = \vec{y}$.* Then $\tilde{a} \vee_d \vec{x} \leq \tilde{a} \vee_L \vec{x} = \vec{y}$ yields that $a \leq y_1 = x_1 \vee p_1$, whence $(a, x_1, p_1) \in L_1^3$ is a perspective triplet by Lemma 6. In particular, this implies $a \vee x_1 = p_1 \vee x_1$. We obtain $b \vee x_1 = p_1 \vee x_1$ similarly. Hence $(a, x_1, b)$ is a perspective triplet in $L_1$, contradicting $a \not\sim b$. This proves the second part of the lemma.

In case of the third part, $\tilde{a} = (a, 0)$ and $\tilde{b} = (0, b)$. Since $\vec{y} \geq \tilde{a} \vee_d \vec{x}$ and $\vec{y} \geq \tilde{b} \vee_d \vec{x}$, we see that $x_1 \neq y_1$ and $x_2 \neq y_2$. Hence $\vec{x} \prec_d \vec{y}$ fails, whence we obtain from (12) that $\vec{p} \not\leq \vec{x}$ and $\vec{y} = \vec{x} \vee_d \vec{p} = \vec{x} \vee_L \vec{p}$. Consequently, $a \leq y_1 = x_1 \vee p_1$ and $b \leq y_2 = x_2 \vee p_2$ , and Lemma 6 shows that both $(a, x_1, p_1) \in L_1^3$ and $(b, x_2, p_2) \in L_2^3$ are perspective triplets, a contradiction. ☐

**Lemma 8.** *Let $L = [L_1 \cup L_2; p_1 = p_2] = \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2)$ be an atomic amalgam of two finite semimodular lattices $L_1$ and $L_2$. Then $J(L) = \{\tilde{a} : a \in J(L_1) \cup J(L_2)\}$.*

Roughly speaking, the lemma asserts that $J(L) = J(L_1) \cup J(L_2)$ modulo canonical embedding. This is what one would expect by the alternative definition of $L$ as the ideal lattice of the chopped lattice $(L_1 \cup L_2; p_1 = p_2)$. Since the paper is based on the $\mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2)$ approach, it seems to be reasonable to give a formal proof. (Another reason is that the proof of Lemma 12 will refer to this proof.)

*Proof of Lemma 8.* Let $a \in J(L_1)$. If $p_1 \not\leq a$, then $\tilde{a} = (a, 0) \in J(L)$ is evident. Assume that $p_1 \leq a$, that is, $\tilde{a} = (a, p_2)$. Let $\vec{b} = (b_1, b_2)$ and $\vec{c} = (c_1, c_2)$ be elements of $L$ such that $\vec{b} < \tilde{a}$ and $\vec{c} < \tilde{a}$. We want to show that $\vec{b} \vee_L \vec{c} < \tilde{a}$.

If we had $b_1 = a$, then $b_2 \geq p_2$ would give $\vec{b} \geq \tilde{a}$, a contradiction. Hence $b_1 < a$. Similarly, $c_1 < a$. We know from $a \in J(L_1)$ that $b_1 \vee c_1 < a$. Since $\vec{b}, \vec{c} < \tilde{a}$, we have $b_2, c_2 \leq p_2$.

If $p_1 \not\leq b_1 \vee c_1$, then $p_1 \not\leq b_1, c_1$ and $\vec{b}, \vec{c} \in L$ yield that $b_2 = c_2 = 0$. Hence $\vec{b} \vee_d \vec{c} = (b_1 \vee c_1, 0) \in L$ gives $b \vee_L \vec{c} = (b_1 \vee c_1, 0) < \tilde{a}$, indeed.

If $p_1 \leq b_1 \vee c_1$, then $\tilde{a} = (a, p_2)$, and $\tilde{a} > (b_1 \vee c_1, p_2) \in L$ together with $\vec{b}, \vec{c} \leq (b_1 \vee c_1, p_2)$ yields that $\tilde{a} > \vec{b} \vee_L \vec{c}$. Thus, $\tilde{a} \in J(L)$. The case of $a \in J(L_2)$ is analogous. So, we have seen that $J(L) \supseteq \{\tilde{a} : a \in J(L_1) \cup J(L_2)\}$.

To show the reverse inclusion, let $\vec{x} = (x_1, x_2) \in J(L)$. Then $x_1 = a_1 \vee \cdots \vee a_k$ and $x_2 = b_1 \vee \cdots \vee b_\ell$ for some $a_1, \ldots, a_k \in J(L_1)$ and $b_1, \ldots, b_\ell \in J(L_2)$; here $k, \ell \geq 0$. Since $a_i \leq x_1$, we have $\tilde{a}_i \leq \tilde{x}_1 \leq \vec{x}$. Similarly, $\tilde{b}_j \leq \vec{y}$. Hence

$$\begin{aligned} \vec{x} &\leq \tilde{a}_1 \vee_d \cdots \vee_d \tilde{a}_k \vee_d \tilde{b}_1 \vee_d \cdots \vee_d \tilde{b}_\ell \\ &\leq \tilde{a}_1 \vee_L \cdots \vee_L \tilde{a}_k \vee_L \tilde{b}_1 \vee_L \cdots \vee_L \tilde{b}_\ell \leq \vec{x} \end{aligned}$$

So, the above inequalities are equalities, and the reverse inclusion follows. □

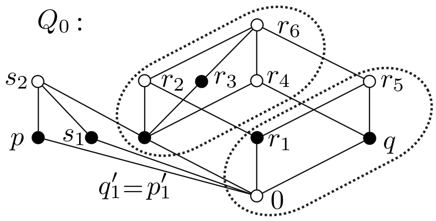


FIGURE 3. $Q_0 = (M_3 \cup T_0; q_1' = p_1')$

Let $M_3$ be the left-hand lattice in Figure 1. (Notice that $M_k$, the modular lattice of length two with $k$ atoms would also do for each $k \geq 3$.) Define the chopped lattice $Q_0 = (M_3 \cup T_0; q_1' = p_1')$, see Figure 3, and let

$$Q^* = [M_3 \cup T_0; q_1' = p_1'] = \operatorname{Cmpv} Q_0.$$

We have canonical embeddings of $M_3$ and $T_0$ into $Q^*$, see (4). In the spirit of these embeddings, we will write $p$ and $q$ instead of $\tilde{p} = (p, 0)$ and $\tilde{q} = (0, q)$. Notice that $Q^*$ consists of 22 elements and, without the previous lemmas, it would be tedious to check its properties in the straightforward way.

**Lemma 9.** $Q^* = (Q^*, \wedge, \vee, p, q)$ *is a perfect gadget.*

*Proof.* Using the fact that $T_0 = (T_0, p_1', q)$ is a basic gadget, is easy to see that the chopped lattice $Q_0$ has only one non-trivial congruence, the congruence denoted by dotted ovals. Using Lemma 3, we conclude that $(Q^*, \wedge, \vee, p, q)$ is a basic gadget. Based on (1), it is evident that $T_0$ is semimodular. By Lemma 5, $Q^*$ is a semimodular lattice. By Lemma 8, $J(Q^*)$, as an order, is (isomorphic to) $J(Q_0)$, the black-filled elements in Figure 3. Hence $Q^*$ is an almost-geometric lattice, that is, (18) holds for $Q^*$. The black-filled elements give (19). Since $p_1' \not\succ q$ in $T_0$, Lemma 7 yields $p \not\succ q$, that is, (17). Hence $(Q^*, \wedge, \vee, p, q)$ is a perfect gadget. □

For $i = 1, 2$, let $L_i$ be a finite semimodular lattice, and let $p_i \not\prec q_i$ be atoms of $L_i$. By the identifications $0_1 = 0_2$, $p_1 = p_2$, $q_1 = q_2$ and $p_1 \vee q_1 = p_2 \vee q_2$, see (16), we obtain a chopped lattice

$$(L_1 \cup L_2; p_1 = p_2, q_1 = q_2),$$

which is called a *biatomic merging* of the lattices $L_1$ and $L_2$. Then the lattice

$$[L_1 \cup L_2; p_1 = p_2, q_1 = q_2] := \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2, q_1 = q_2) \tag{20}$$

is called a *biatomic amalgam* of $L_1$ and $L_2$. Let us emphasize that *this terminology supposes* that $L_1$ and $L_2$ are *finite semimodular* lattices and $p_i \not\prec q_i$ for $i = 1, 2$.

As usual, $\tilde{x}$ denotes the image of $x \in L_i$ under the canonical $L_i \mapsto [L_1 \cup L_2; p_1 = p_2, q_1 = q_2]$ embedding. For example, if $a \in A(L_1) \setminus \{p_1\}$, then $\tilde{a} = (a, 0)$. However, $\tilde{p}_1 = \vec{p} = (p_1, p_2) = \tilde{p}_2$. Sometimes we identify $x$ and $\tilde{x}$. In the next lemma, the covering relation in $L$, $L_1 \times L_2$, $L_p$ and $L_q$ is denoted by $\prec_L$, $\prec_d$, $\prec_p$ and $\prec_q$, respectively. Analogous notation applies for the join operation.

**Lemma 10.** *Let $L = [L_1 \cup L_2; p_1 = p_2, q_1 = q_2]$, $L_p = [L_1 \cup L_2; p_1 = p_2]$ and $L_q = [L_1 \cup L_2; q_1 = q_2]$; note that $L = L_p \cap L_q$. Let $\vec{e}, \vec{f} \in L$. Then $\vec{e} \prec_L \vec{f}$ iff* exactly one *of the following three possibilities holds:*

$$\vec{e} \prec_d \vec{f}; \tag{21}$$

$$\vec{e} \prec_p \vec{f}, \ \ \vec{e} \not\geq \vec{p} \ \text{ and } \ \vec{f} = \vec{e} \vee_d \vec{p} = \vec{e} \vee_L \vec{p}; \tag{22}$$

$$\vec{e} \prec_q \vec{f}, \ \ \vec{e} \not\geq \vec{q} \ \text{ and } \ \vec{f} = \vec{e} \vee_d \vec{q} = \vec{e} \vee_L \vec{q}. \tag{23}$$

*Further,*

$$\text{if } \vec{p} \not\leq \vec{e} \text{ and } \vec{p} \not\leq \vec{f}, \text{ or } \vec{p} \leq \vec{e} \text{ and } \vec{p} \leq \vec{f}, \text{ then } \vec{e} \prec_L \vec{f} \text{ iff } \vec{e} \prec_q \vec{f}, \tag{24}$$

$$\text{if } \vec{q} \not\leq \vec{e} \text{ and } \vec{q} \not\leq \vec{f}, \text{ or } \vec{q} \leq \vec{e} \text{ and } \vec{q} \leq \vec{f}, \text{ then } \vec{e} \prec_L \vec{f} \text{ iff } \vec{e} \prec_p \vec{f}. \tag{25}$$

According to Lemma 10, each covering pair $\vec{e} \prec_L \vec{f}$ in $L$ has a *unique type*. Namely, if (21), (22) or (23) holds, then we will say that $\vec{e} \prec_L \vec{f}$ is of type $d$, type $p$ or type $q$, respectively. Notice that, by (12) applied to $L_p$, (22) is equivalent to the conjunction of $\vec{e} \not\geq \vec{p}$ and $\vec{f} = \vec{e} \vee_d \vec{p} = \vec{e} \vee_L \vec{p}$. Similarly, $\vec{e} \prec_q \vec{f}$ can be omitted from (23).

*Proof of Lemma 10.* The second part, stating (24) and (25), is evident.

If (22), then $\vec{e} \prec_L \vec{f}$ follows from $L \subseteq L_p$ and (12). Similarly, each of (21) and (23) implies $\vec{e} \prec_L \vec{f}$ evidently.

The conjunction of (22) and (23) contradicts to $p_1 \not\prec q_1$ (and also to $p_2 \not\prec q_2$). Hence it is easy to see that no two of the conditions (21), (22) and (23) can hold simultaneously.

Next, we assume that $\vec{e} \prec_L \vec{f}$. We also assume that (21) fails. We want to show that (22) or (23) holds. If $\vec{p} \leq \vec{e}$ or $\vec{p} \not\leq \vec{f}$, then (24), combined with (12) for $L_q$, yields (23). Similarly, if $\vec{q} \leq \vec{e}$ or $\vec{q} \not\leq \vec{f}$, then (25), combined with (12) for $L_p$, yields (22). Therefore, we can assume that

$$\vec{p} \not\leq \vec{e}, \quad \vec{q} \not\leq \vec{e}, \quad \vec{p} \leq \vec{f} \quad \text{and} \quad \vec{q} \leq \vec{f}.$$

We can also assume that $\vec{e} \not\prec_p \vec{f}$, since otherwise (12) for $L_p$ would imply (22). Since $\vec{e} \prec_p \vec{e} \vee_d \vec{p}$ by (12) for $L_p$, we see that $\vec{e} \vee_d \vec{p} \neq \vec{f}$. Hence $\vec{e} < \vec{e} \vee_d \vec{p} < \vec{f}$. We obtain from $\vec{e} \prec_L \vec{f}$ that $\vec{e} \vee_d \vec{p}$ is not in $L$. However, it is clearly in $L_p$, so $\vec{e} \vee_d \vec{p} \notin L_q$. Hence there are $i$ and $j$ such that $\{i, j\} = \{1, 2\}$, $e_i \vee p_i \geq q_i$ and $e_j \vee p_j \not\geq q_j$. Therefore, $(p_i, e_i, q_i)$ is a perspective triple by Lemma 6, a contradiction. □

**Lemma 11.** *Every biatomic amalgam is a semimodular lattice.*

*Proof.* Motivated by (1), we consider a lower covering square $H$ in $L$. We use the notations given in (13) and (14). In particular, $\vec{w} := \vec{y} \vee_d \vec{z}$. If $\vec{p} \leq \vec{x}$ or $\vec{p} \not\leq \vec{v}$, then $H$ is a covering square in $L_q$ by (24) and Lemma 5, whence $H$ is a covering square in $L$ as well. Since the same argument works for $q$ instead of $p$, we can assume that

$$\vec{p} \not\leq \vec{x}\,, \quad \vec{q} \not\leq \vec{x}\,, \quad \vec{p} \leq \vec{v}\,, \text{ and } \vec{q} \leq \vec{v}\,.$$

There are nine cases according to the types of $\vec{x} \prec_L \vec{y}$ and $\vec{x} \prec_L \vec{z}$. However, we have to deal only with the following four cases, since the rest are obviously settled by $p$-$q$ and $y$-$z$ symmetries.

*Case* 1: $\vec{x} \prec_d \vec{y}$ *and* $\vec{x} \prec_d \vec{z}$. Assume the $x_1 \neq y_1$. (The other case, $x_2 \neq y_2$ is quite the same.). Then $x_1 \prec y_1$ and $\vec{y} = (y_1, x_2)$. Since $\vec{y} \in L$, we have $p_1, q_1 \not\leq y_1$.

*Subcase* 1.1: $z_2 \neq x_2$. Then $\vec{z} = (x_1, z_2) \in L$ and $x_2 \prec z_2$ . Since $\vec{z} \in L$, we see that $p_2, q_2 \not\leq z_2$. Hence $\vec{w} = (y_1, z_2) \in L$, so $\vec{v} = \vec{w}$. Since $\vec{y}, \vec{z} \prec_d \vec{w}$, we obtain that $\vec{y}, \vec{z} \prec_L \vec{v}$. Hence $H$ is a covering square in $L$, as desired.

*Subcase* 1.2: $z_2 = x_2$. Then $x_1 \prec z_1 \neq y_1$, $\vec{z} = (z_1, x_2)$, $\vec{w} = (y_1 \vee z_1, x_2)$, $y_1 \prec w_1$ and $z_1 \prec w_1$.

*Sub-subcase* 1.2.1: $p_1 \not\leq w_1$ *and* $q_1 \not\leq w_1$. Then $\vec{w} \in L$, so $\vec{v} = \vec{w}$, and $\vec{y}, \vec{z} \prec_d \vec{w}$ implies that $H$ is an upper covering square in $L$.

*Sub-subcase* 1.2.2: $p_1 \leq w_1$ *and* $q_1 \leq w_1$. Then $y_1 \prec w_1$ and $y_1 \prec y_1 \vee p_1 \leq w_1$ gives $y_1 \vee p_1 = w_1$. Since the same holds for $q_1$, we obtain that $(p_1, y_1, q_1)$ is a perspective triplet, which contradicts $p_1 \not\sim q_1$. Hence this sub-subcase is excluded.

*Sub-subcase* 1.2.3: $p_1 \leq w_1$ *and* $q_1 \not\leq w_1$. Then we obtain $w_1 = y_1 \vee p_1 = z_1 \vee p_1$ from $y_1, z_1 \prec w_1$. Let $\vec{u} := \vec{w} \vee_d \vec{p} = (w_1, x_2 \vee p_2)$.

Firstly, assume that $\vec{u} \in L$. Then, since $\vec{u} = \vec{w} \vee_d \vec{p} \leq \vec{v}$, we have that $\vec{v} = \vec{u}$. Hence $\vec{q} \leq \vec{v}$ gives that $q_2 \leq v_2 = u_2 = x_2 \vee p_2$, and Lemma 6 implies that $(p_2, x_2, q_2)$ is a perspective triplet, a contradiction.

Secondly, assume that $\vec{u} \notin L$. Then $\vec{u} \notin L_q$. Since $q_1 \not\leq w_1 = u_1$, we see that $q_2 \leq u_2 = x_2 \vee p_2$. Hence $(p_2, x_2, q_2)$ is a perspective triplet by Lemma 6 again, a contradiction. Thus, Sub-subcase 1.2.3 is excluded

*Sub-subcase* 1.2.4: $p_1 \not\leq w_1$ *and* $q_1 \leq w_1$. By $\vec{p}$-$\vec{q}$ symmetry, the argument for Sub-subcase 1.2.3 excludes this sub-subcase as well.

*Case* 2: $\vec{x} \prec_d \vec{y}$ *and* $\vec{x} \prec_p \vec{z}$. Let, say, $y_1 \neq x_1$. Then $x_1 \prec y_1$ and $\vec{y} = (y_1, x_2)$. Since $\vec{z} = (x_1 \vee p_1, x_2 \vee p_2)$, we get that $\vec{w} = (y_1 \vee p_1, x_2 \vee p_2)$. The covering $\vec{x} \prec_L \vec{z}$ is not of type $q$ by Lemma 10, so $q_2 \not\leq z_2 = w_2$. From $y_2 = x_2$ and $\vec{y} \in L$ we obtain that $p_1 \not\leq y_1$ and $q_1 \not\leq y_1$. If we had $q_1 \leq w_1 = y_1 \vee p_1$, then $(p_1, y_1, q_1)$ would be a perspective triplet by Lemma 6. Hence $q_1 \not\leq w_1$, and we see that $\vec{w} \in L$, so $\vec{v} = \vec{w}$. The semimodularity of $L_1 \times L_2$ and $\vec{x} \prec_d \vec{y}$ yields that $\vec{z} \prec_d \vec{w} = \vec{v}$. Part (22) of Lemma 10 implies $\vec{y} \prec_L \vec{y} \vee_d \vec{p} = \vec{w} = \vec{v}$. Hence $H$ is a covering square in $L$.

*Case* 3: $\vec{x} \prec_p \vec{y}$ *and* $\vec{x} \prec_p \vec{z}$. This would imply $\vec{p} \le \vec{x}$, so this case has been excluded.

*Case* 4: $\vec{x} \prec_p \vec{y}$ *and* $\vec{x} \prec_q \vec{z}$. Then $\vec{y} = \vec{x} \vee_d \vec{p}$, $\vec{z} = \vec{x} \vee_d \vec{q}$, and $\vec{w} = \vec{y} \vee_d \vec{z} = \vec{y} \vee_d \vec{q}$ and, similarly, $\vec{w} = \vec{z} \vee_d \vec{p}$. Since $\vec{w} \in L$, we conclude that $\vec{v} = \vec{w}$. The covering $\vec{x} \prec_L \vec{z}$ is not of type $p$ and $\vec{p} \not\le \vec{x}$, so we get that $\vec{p} \not\le \vec{z}$. Similarly, $\vec{q} \not\le \vec{y}$. Hence (22) and (23) yield that $\vec{z} \prec_L \vec{w} = \vec{u}$ and $\vec{y} \prec_L \vec{w} = \vec{u}$, respectively. This means that $H$ is an upper covering square in $L$. □

The "biatomic" counterpart of Lemma 8 will need condition (26). Note that $\tilde{p}_1 = \tilde{p}_2 = (p_1, p_2)$ and $\tilde{q}_1 = \tilde{q}_2 = (q_1, q_2)$ will be denoted by $\vec{p}$ and $\vec{q}$, respectively.

**Lemma 12.** *Let* $L = [L_1 \cup L_2; p_1 = p_2, q_1 = q_2] = \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2, q_1 = q_2)$ *be a biatomic amalgam. Assume that*

$$J(L_i) \cap \uparrow p_i = \{p_i\} \text{ and } J(L_i) \cap \uparrow q_i = \{q_i\} \text{ for } i = 1, 2. \tag{26}$$

*Then*

- $J(L) = \{\tilde{a} : a \in J(L_1) \cup J(L_2)\}$.
- *If, in addition,* $L_1$ *and* $L_2$ *are almost-geometric, then* $L$ *is almost geometric and*

$$J(L) \cap \uparrow \vec{p} = \{\vec{p}\} \text{ and } J(L) \cap \uparrow \vec{q} = \{\vec{q}\}. \tag{27}$$

*Proof.* If $a = p_1$, then $\tilde{a} = (p_1, p_2) \in J(L)$ is trivial. In fact, $\tilde{a} \in A(L)$. Similarly, $\tilde{q}_1 \in J(L)$. Let $a \in J(L_1) \setminus \{p_1, q_1\}$. Then

$$\tilde{a} = (a, 0) \tag{28}$$

by (26), whence $\tilde{a} \in J(L)$ again. Similarly, $\tilde{a} \in J(L)$ for every $a \in J(L_2)$. Hence $J(L) \supseteq \{\tilde{a} : a \in J(L_1) \cup J(L_2)\}$.

Before dealing with the reverse inclusion, we show that

$$\text{if } \vec{x} = (x_1, x_2) \in L, \text{ then } \tilde{x}_1 \le \vec{x} \text{ and } \tilde{x}_2 \le \vec{x}. \tag{29}$$

Indeed, combining the definition of a biatomic amalgam with (4), we obtain that the canonical embedding of $L_1$ into $L$ sends $x_1$ to

$$\tilde{x}_1 := \begin{cases} (x_1, 0) & \text{if } p_1 \not\le x_1 \text{ and } q_1 \not\le x_1 \\ (x_1, p_2) & \text{if } p_1 \le x_1 \text{ and } q_1 \not\le x_1 \\ (x_1, q_2) & \text{if } p_1 \not\le x_1 \text{ and } q_1 \le x_1 \\ (x_1, p_2 \vee q_2) & \text{if } p_1 \le x_1 \text{ and } q_1 \le x_1 \end{cases}.$$

This implies (29). Armed with (29), the reverse inclusion follows exactly the same way as in the proof of Lemma 8.

Finally, the second part of the lemma is an evident consequence of the first part and (28). □

**Lemma 13.** *Let* $L = [L_1 \cup L_2; p_1 = p_2, q_1 = q_2] = \mathrm{Cmpv}(L_1 \cup L_2; p_1 = p_2, q_1 = q_2)$ *be a biatomic amalgam. For* $i = 1, 2$*, let* $B_i \subseteq A(L_i)$ *be a set of pairwise non-perspective atoms such that* $\{p_i, q_i\} \subseteq B_i$.

- *If* $a, b \in B_i$ *and* $a \ne b$*, then* $\tilde{a} \not\sim \tilde{b}$ *in* $L$.
- *If* $a \in B_1$*,* $b \in B_2$ *and* $\tilde{a} \ne \tilde{b}$*, then* $\tilde{a} \not\sim \tilde{b}$ *in* $L$.

*Proof.* Assume, by way of contradiction, that $(\tilde{a}, \vec{x}, \tilde{b}) \in L^3$ is a perspective triplet. Let $\vec{y} = \tilde{a} \vee_L \vec{x} = \tilde{b} \vee_L \vec{x}$. Since $\tilde{a}, \tilde{b} \in A(L)$ and $L$ is semimodular by Lemma 11, $\vec{x} \prec_L \vec{y}$. Since $\tilde{p}_1 = \tilde{p}_2 = (p_1, p_2) = \vec{p}$ and $\tilde{q}_1 = \tilde{q}_2 = \vec{q}$, there are only four essentially different cases.

*Case $a = p_1$ and $b = q_1$.* Then, in $L$, we have $\tilde{a} = \tilde{p}_1 = (p_1, p_2) = \vec{p}$ and $\tilde{b} = (q_1, q_2) = \vec{q}$. Since $\tilde{p} \not\leq \vec{x}$ and $\vec{p} \leq \vec{y}$, the covering $\vec{x} \prec_L \vec{y}$ is of type $p$, see Lemma 10. Similarly, it is of type $q$, contradicting the uniqueness of types.

*Case $a = p_1$ and $b \in L_1 \setminus \{p_1, q_1\}$.* Then $\tilde{b} = (b, 0) \in L$. Since type $p$ is the only possibility for $\vec{x} \prec_L \vec{y}$, we have $\vec{y} = (x_1 \vee p_1, x_2 \vee p_2)$. Hence $b \leq x_1 \vee p_1$, and Lemma 6 yields that $(b, x_1, p_1)$ is a perspective triplet. This contradicts $b \not\sim p_1$.

*Case $a, b \in L_1 \setminus \{p_1, q_1\}$.* Then $\tilde{a} = (a, 0)$ and $\tilde{b} = (b, 0)$. If the covering $x \prec_L y$ is of type $d$, then $(a, x_1, b) \in L_1^3$ is a perspective triplet, a contradiction. Hence we can assume that this covering is of type $p$, that is, $\vec{y} = (x_1 \vee p_1, x_2 \vee p_2)$. Then $b \leq y_1 = x_1 \vee p_1$, and Lemma 6 leads to the perspectivity of the triplet $(b, x_1, p_1)$, a contradiction again.

*Case $a \in L_1$ and $b \in L_2$.* Then $\tilde{a} = (a, 0)$ and $\tilde{b} = (0, b)$. Since $\tilde{a} \vee_d \vec{x} = (a \vee x_1, x_2)$ and $\tilde{b} \vee_d \vec{x} = (x_1, b \vee x_2)$ are incomparable elements of $L_1 \times L_2$ between $\vec{x}$ and $\vec{y}$, the covering $\vec{x} \prec_L \vec{y}$ cannot be of type $d$. Hence, say, $\vec{x} \prec_p \vec{y}$, whence $(a, x_1, p_1) \in L_1^3$ is a perspective triplet by Lemma 6. This contradicts $a \not\sim p_1$. ☐

**Lemma 14.** *Assume that $S_i$ is a finite semimodular lattice and $p_i, q_i \in A(S_i)$ such that $p_i \not\sim q_i$, for all meaningful $i$. With these assumptions, let $C_m$ denote the chopped lattice defined in* (6).

- *With the notations prior to* (6)*, there are $a, b \in \{a_1, \ldots, a_n\}$ such that*

$$\operatorname{Cmpv} C_{m+1} \cong [S_{m+1} \cup \operatorname{Cmpv} C_m; p_{m+1} = a, q_{m+1} = b].$$

- *The 1960-amalgam given in* (7) *is a semimodular lattice.*

*Proof.* Let $L_m = \operatorname{Cmpv} C_m$ denote the lattice given in (7); all the notations between (5) and (7) will be in effect. For $a_i \in M_0^\wedge$, the canonical embedding of $M_0^\wedge$ into $L_m$ allows us to say $a_i \in L_m$; however, we often use $\tilde{a}_i \in L_m$ to denote the same element. Let $n$, the number of atoms of $M_0^\wedge$, be fixed.

We prove the theorem by induction on $m$. The induction hypothesis is that

$H(m)$: $L_m$ is a semimodular lattice, and $a_1, \ldots, a_n$ are pairwise non-perspective atoms in $L_m$.

Although this induction hypothesis seems to work only for the second part of the lemma, we point out that (30) will settle the first part as well.

The initial step of the induction, $m = 0$, is evident, since $L_0$ is the Boolean lattice with $A(L_0) = \{a_1, \ldots, a_m\}$.

Let us assume that $H(m)$ holds. Firstly, we fix some notations. Let $C' = C_m$ and $L' = L_m = \operatorname{Cmpv} C'$. By the induction hypothesis, $L'$ is a semimodular lattice and $a_1, \ldots, a_n$ are pairwise non-perspective atoms of $L'$. Let $G_{m+1} = G_m \cup \{(a, b)\}$.

Of course, $a, b \in \{a_1, \dots, a_n\}$). Let

$$U = \{u \in \operatorname{Max} C' : a \le u\},\ V = \{v \in \operatorname{Max} C' : b \le v\},\ \text{and}$$
$$W = \{w \in \operatorname{Max} C' : a \not\le w,\ b \not\le w\}.$$

Then $\operatorname{Max} C' = U \cup V \cup W$, see Figure 4. (Figure 4 tries to depict the general case, some of its parts may be missing.) Notice that each $x \in \operatorname{Max} C'$ is either the top element of some $S_i$ occurring in (6) or $x \in \{a_1, \dots, a_n\}$. Since $(a, b) \notin G_m$, we see that $U$, $V$ and $W$ are pairwise disjoint. We denote the greatest element of $S := S_{m+1}$ by $g$, and $p_{m+1}$ and $q_{m+1}$ by $p$ and $q$. Remember that $p$ and $q$ are non-perspective atoms of $S$. Let $C = C_{m+1}$, see Figure 5, and $L = L_{m+1} = \operatorname{Cmpv} C$. Notice that $C = C' \cup S$ with the identifications $0_{C'} = 0_S$, $a = p$ and $b = q$. We have to show $H(m+1)$.

Let $\vec{u} = (u_1, u_2 \dots)$, $\vec{v} = (v_1, , v_2, \dots)$ and $\vec{w} = (w_1, w_2, \dots)$ be fixed enumerations of $U$, $V$ and $W$, respectively. (Notice that some of these vectors can be empty.) Then $L' = \operatorname{Cmpv} C'$ consists of compatible vectors $(\vec{x}, \vec{y}, \vec{z})$ where $x_i \le u_i$, $y_j \le v_j$ and $z_k \le w_k$ for all meaningful $i, j, k$, see Figure 4. Consider the biatomic amalgam

$$K = [S \cup L'; p = a, q = b] = \operatorname{Cmpv}(S \cup L'; p = a, q = b).$$

It suffices to show that

$$\begin{aligned} &\text{there exists a lattice isomorphism } \varphi\colon K \to L \\ &\text{such that } \varphi \text{ acts identically on } M_0^\wedge. \end{aligned} \tag{30}$$

Indeed, in virtue of Lemmas 11 and 13, $H(m)$ and (30) imply $H(m+1)$. Apart from $a$-$b$-symmetry, there are three cases.

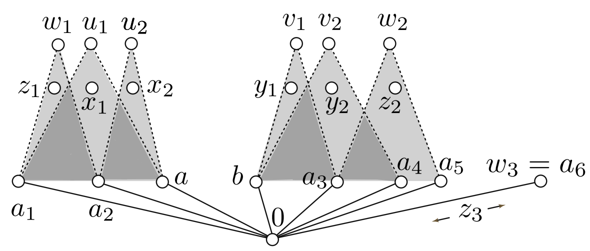


FIGURE 4. The chopped lattice $C' = C_m$ in Case 1

*Case* 1: *$a \notin U$ and $b \notin V$.* We know that $K$ consists of compatible vectors

$$\vec{\alpha} = (t, \vec{\gamma}) = (t, (\vec{x}, \vec{y}, \vec{z})), \quad \text{where } t \in S \text{ and } \vec{\gamma} = (\vec{x}, \vec{y}, \vec{z}) \in L'; \tag{31}$$

see Figure 4 for an illustration. (Note that, say, $\downarrow w_1 \cap \downarrow u_1 = \{0, a_1\}$; the darker "common area" indicates disjointness.) On the other hand, $\operatorname{Max} C = \{g\} \cup U \cup V \cup W$, see Figure 5. Hence $L = \operatorname{Cmpv} C$ consists of compatible vectors

$$\vec{\beta} = (t, \vec{x}, \vec{y}, \vec{z}) \tag{32}$$

where $t \le g$, $x_i \le u_i$, $y_j \le v_j$ and $z_k \le w_k$; see Figure 5. Keeping (31) and (32) in

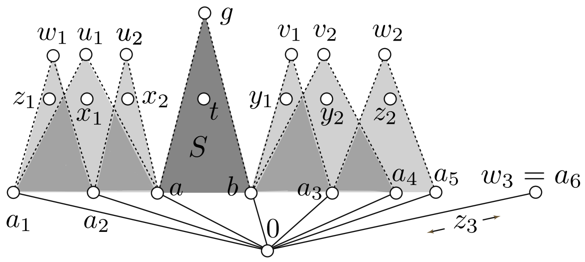


FIGURE 5. The chopped lattice $C = C_{m+1}$ in Case 1

mind, we define

$$\varphi\colon K \to L,\ \vec{\alpha} \mapsto \vec{\beta} \quad \text{and} \quad \psi\colon L \to K,\ \vec{\beta} \mapsto \vec{\alpha}.$$

We have to show that $\varphi$ maps into $L$ and $\psi$ maps into $K$, that is, both $\varphi$ and $\psi$ send compatible vectors to compatible vectors. If this is shown, then $\varphi$ and $\psi$ are clearly lattice isomorphisms, since they are reciprocal order-preserving bijections.

Assume that $\vec{\alpha}$ is compatible, that is, $\vec{\alpha} \in \operatorname{Cmpv} C$. Since $\vec{\gamma} \in L'$, all components of $\vec{\beta}$ but $t$ are evidently "compatible" in the sense of (2), see also (3). (Indeed, for example, consider $x_i$ and $z_k$. If $u_i \wedge w_k = 0$, then $x_i$ and $z_k$ are always compatible. Otherwise, $u_i \wedge w_k$ is an atom $a_\ell$ of $C'$ that belongs to $M_0^\wedge$. Since $\vec{\gamma} \in L'$, either $a_\ell \le x_i, z_k$ or $a_\ell \nleq x_i, z_k$, and $x_i$ and $z_k$ are compatible components of $\vec{\beta}$ in both cases.)

We have to show that $t$ is compatible with the rest of components of $\vec{\beta}$. Since $g \wedge w_k = 0$, it is clear that $t$ and $z_k$ are compatible.

Consider $t$ and $x_i$, and remember that $g \wedge u_i = a$. Firstly, if $a \le t$ in $C$, then $p \le t$ in $S$. Since $\vec{\alpha}$ is compatible, $\tilde{a} \le \vec{\gamma}$ in $L'$. Here $\tilde{a} = (a, \dots, a, \vec{0}, \vec{0})$ by (4). Hence $a \le x_i$, as desired. Secondly, if $a \nleq t$ in $C$, then $p \nleq t$ in $S$. The compatibility of $\vec{\alpha}$ yields that $\tilde{a} \nleq \vec{\gamma}$ in $L'$. Hence $a \nleq x_\ell$ for some $\ell$. But $\vec{\gamma} \in L'$ implies that $x_i$ and $x_\ell$ are compatible, whence $a \nleq x_i$, as desired. So, $t$ and $x_i$ are compatible.

Since $a$ and $b$ play symmetric roles, $t$ and $y_j$ are compatible as well, and we conclude that $\vec{\beta}$ is compatible.

Conversely, assume that $\vec{\beta}$ is compatible. Then $\vec{\gamma} = (\vec{x}, \vec{y}, \vec{z})$ is clearly compatible, so it belongs to $L'$. By symmetry, it suffices to check the compatibility of $\vec{\alpha}$ "with respect to $p = a$". Firstly, suppose that $p \le t$ in $S$. Then $a \le t$ in $C$, whence $\beta \in L$ implies that $a \le x_i$ for all meaningful $i$. So $\tilde{a} = (a, \dots, a, \vec{0}, \vec{0}) \le \vec{\gamma}$, as desired. Secondly, suppose that $p \nleq t$ in $S$. Then $a \nleq t$ in $C$, whence $\beta \in L$ implies that $a \nleq x_i$ for all meaningful $i$. So $\tilde{a} = (a, \dots, a, \vec{0}, \vec{0}) \nleq \vec{\gamma}$. This shows that $\vec{\alpha} \in K$, indeed.

*Case* 2: $a \in U$ *and* $b \notin V$. Then $U = \{a\}$, and $K$ consists of compatible vectors

$$\vec{\alpha} = (t, \vec{\gamma}) = (t, (x, \vec{y}, \vec{z})), \quad \text{where } t \in S \text{ and } \vec{\gamma} = (x, \vec{y}, \vec{z}) \in L', \tag{33}$$

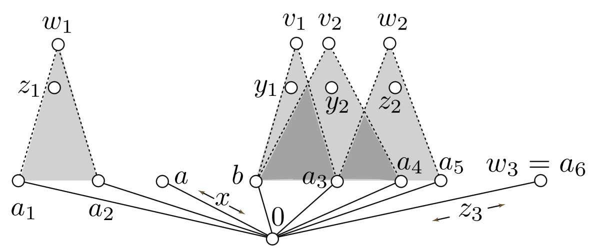


FIGURE 6. The chopped lattice $C' = C_m$ in Case 2

see Figure 6. Here $x \in \{0, a\}$. Since $\operatorname{Max} C = \{g\} \cup V \cup W$, we know that $L = \operatorname{Cmpv} C$ consists of compatible vectors

$$\vec{\beta} = (t, \vec{y}, \vec{z}), \tag{34}$$

see Figure 7. Let us define

$$\varphi\colon K \to L,\ \vec{\alpha} \mapsto \vec{\beta}$$
$$\psi\colon L \to K,\ \vec{\beta} \mapsto \vec{\alpha},\ \text{where } x = a \wedge t \text{ (in } C).$$

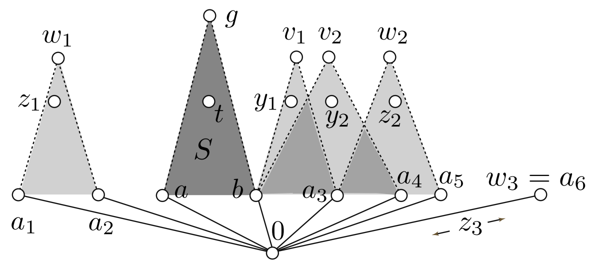


FIGURE 7. The chopped lattice $C = C_{m+1}$ in Case 2

Assume that $\vec{\alpha}$ is compatible; we have to show that so is $\vec{\beta}$. The compatibility of the $(\vec{y}, \vec{z})$ part of $\vec{\beta}$ is clear. So is the compatibility of $t$ and $z_k$. Observe that $b \leq t$ in $C$ iff $q \leq t$ in $S$ iff $\tilde{b} = (0, b, \dots, b, \vec{0}) \leq \vec{\gamma}$ iff $b \leq y_j$ for all $j$. However, $b \leq y_j$ for all $j$ iff $b \leq y_j$ for some $j$, since $\vec{\gamma} \in L'$. This shows that $\vec{\beta}$ is compatible.

Conversely, assume that $\vec{\beta}$ is compatible. Clearly, no matter if $x = a \wedge t$ is 0 or $a$, the vector $\vec{\gamma}$ is compatible. Since

$$\begin{aligned} p \leq t \text{ (in } S) &\iff a \leq t \text{ (in } C) \iff a = a \wedge t \text{ (in } C) \\ &\iff a = x \text{ (in } C) \iff a \leq x \text{ (in } C') \\ &\iff \tilde{a} = (a, \vec{0}, \vec{0}) \leq (x, \vec{y}, \vec{z}) = \vec{\gamma} \text{ (in } L'), \end{aligned} \tag{35}$$

we conclude that $t$ and $\vec{\gamma}$ are compatible with respect to $p = a$. Further, $q \leq t$ in $S$ iff $b \leq t$ in $C$ iff $b \leq y_j$ for all $j$ iff $\tilde{b} = (0, b, \dots, b, \vec{0}) \leq \vec{\gamma}$, which shows that $t$

and $\vec{\gamma}$ are compatible with respect to $q = b$ as well. Hence $\vec{\alpha}$ is compatible, that is, $\vec{\alpha} \in K$, indeed.

Finally, to derive that $\varphi$ is injective (equivalently, $\varphi$ is the inverse of $\psi$), we have to show that if $\vec{\alpha}$ in (33) is compatible, then $x$ is determined by $t$. But this is evident, since $x \in \{0, a\}$, and $p \le t$ in $S$ iff $(a, \vec{0}, \vec{0}) = \tilde{a} \le \vec{\gamma} = (x, \vec{y}, \vec{z})$.

*Case* 3: $a \in U$ *and* $b \in V$. Then $U = \{a\}$, $V = \{b\}$, and $K$ consists of compatible vectors

$$\vec{\alpha} = (t, \vec{\gamma}) = (t, (x, y, \vec{z})), \quad \text{where } t \in S \text{ and } \vec{\gamma} = (x, y, \vec{z}) \in L'. \tag{36}$$

Here $x \in \{0, a\}$ and $y \in \{0, b\}$. Since $\operatorname{Max} C = \{g\} \cup W$, we know that $L = \operatorname{Cmpv} C$ consists of compatible vectors

$$\vec{\beta} = (t, \vec{z}) \tag{37}$$

Let us define

$$\varphi\colon K \to L,\ \vec{\alpha} \mapsto \vec{\beta}$$

$$\psi\colon L \to K,\ \vec{\beta} \mapsto \vec{\alpha},\ \text{where } x = a \wedge t \text{ and } y = b \wedge t \text{ (in } C);$$

Let $\vec{\alpha} \in K$. Since $t$ and $z_i$ are always compatible, $\beta \in L$ is evident. Moreover, like in Case 2, $t$ in (36) determines $x$ and $y$.

Conversely, let $\vec{\beta} \in L$. Then $\vec{\gamma} \in L'$. An argument analogous to (35) together with $a$-$b$ symmetry gives that $t$ and $\vec{\gamma}$ are compatible, whence $\vec{\alpha} \in K$. ☐

**Lemma 15.** *Assume that $(S_i, \wedge, \vee, p_i, q_i)$ is a perfect gadget for each $i \in \{1, \dots, m\}$. Then the 1960-amalgam $L_m$ of these $S_i$, see* (7)*, is an almost-geometric lattice. Moreover, $\uparrow a_i \cap J(L_m) = \{a_i\}$ for $i = 1, \dots, n$.*

*Proof.* The case $m = 0$, the finite Boolean algebra case, is evident. Suppose the statement holds for $L_m = \operatorname{Cmpv} C_m$. Combining the first part of Lemma 14 with Lemma 12, we obtain the statement for $L_{m+1}$. ☐

*Proof of Theorem 1.* Let $G := \operatorname{Con}^{-\mathbf{1}}(D, Q^*)$, see (8) and Lemma 9. Then $G$ is a finite almost-geometric lattice by Lemma 15, and $D \cong \operatorname{Con} G$ by Lemma 4. ☐

## 3. Historical remarks

A classical theorem of R.P. Dilworth [2] states that each finite distributive lattice is isomorphic to $\operatorname{Con} L$ for an appropriate finite lattice. Since 1962, when the first proof of the above theorem was published by G. Grätzer and the second author [10], very many stronger results have been proved. Grätzer [4] gives an excellent survey up to 2005, so we mention only a few milestones, focusing only on those results that yield an appropriate $L$ with some nice additional properties.

The proof in [10] produces a sectionally complemented $L$. Atomic amalgams (of finitely many lattices) play an important role in [10]. According to a nontrivial result of G. Grätzer, H. Lakser and M. Roddy [6], "non-atomic" amalgams need not preserve sectional complementedness. In our case, even less amalgams are appropriate, because $[T_0 \cup T_0; p_1' = p_1']$ is clearly not an almost-geometric lattice. Unfortunately, the present result cannot be combined with [10], since a finite sectionally complemented almost-geometric lattice is necessarily geometric and, therefore, simple.

Let $n = |J(D)|$. Another nice property of $L$ is that $|L|$, the size of $L$, is small compared with $n$. The present paper and [10] produce $L$ with exponential size. The best construction yields a planar $L$ of size $O(n^2)$, see [7]. G. Grätzer, H. Lakser and N. Zaguia [9] proved that we cannot do essentially better if planarity is dropped.

If we require semimodularity, then $L$ of size $O(n^3)$ can be constructed, see [8]. Unless some additional property like that in Grätzer and Knapp [5] is added, we do not know if $O(n^3)$ is optimal for the semimodular case.

Infinite distributive lattices are much less pleasant. Solving a very old problem, F. Wehrung [13] has recently constructed an infinite distributive lattice $D$ such that $D \cong \operatorname{Con} L$ holds for no lattice $L$. A smaller but still infinite $D$ is given by P. Ružička[12].

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