REPRESENTING CONGRUENCE LATTICES OF LATTICES WITH PARTIAL UNARY OPERATIONS AS CONGRUENCE LATTICES OF LATTICES. II. INTERVAL ORDERING

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Abstract. In Part I of this paper, we introduced a method of making two isomorphic intervals of a bounded lattice congruence equivalent. In this paper, we make one interval dominate another one.

Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let φ be a homomorphism of $[a, b]$ *onto* $[c, d]$. We construct a bounded (convex) extension K of L such that a congruence Θ of L has an extension to K iff $x \equiv y \ (\Theta)$ implies that $x\varphi \equiv y\varphi \ (\Theta)$, for $a \leq x \leq y \leq b$, in which case, Θ has a unique extension to K.

This result presents a lattice K whose congruence lattice is derived from the congruence lattice of L in a new way, different from the one presented in Part I.

The main technical innovation is the 2/3-*Boolean triple construction,* which owes its origin to the Boolean triple construction of G. Grätzer and F. Wehrung.

1. Introduction

To keep this paper short, we assume that the reader is familiar with [6], Part I of this paper. Recall that the lattice K is an *extension* of the lattice L, if L is a sublattice of K. The lattice K is a *convex extension* of the lattice L, if L is a convex sublattice of K. A *convex embedding* is defined analogously.

In Part I, we constructed a "magic wand"—as a (convex) extension—that will force that $a \equiv b$ be equivalent to $c \equiv d$ in a bounded lattice L. In this paper, we construct a "one-directional magic wand": $a \equiv b$ implies that $c \equiv d$.

1.1. **One surjective homomorphism.** Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, let φ be a homomorphism of $[a, b]$ *onto* $[c, d]$. We can consider φ as a partial unary operation. Let us call a congruence Θ of L a $\vec{\varphi}$*-congruence* iff Θ satisfies the Substitution Property with respect to the partial unary operation φ, that is, $x \equiv y \ (\Theta)$ implies that $x\varphi \equiv y\varphi \ (\Theta)$, for all x, $y \in [a, b]$. (The symbol $\rightarrow$ on top of φ signifies that the partial operation goes only one way.) Let $L_{\vec{\varphi}}$ denote the partial algebra obtained from L by adding the partial operation φ.

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Thus, a congruence relation of $L_{\vec{\varphi}}$ is the same as a $\vec{\varphi}$-congruence of L. We call an extension K of L a $\vec{\varphi}$-*congruence-preserving extension* of L, if a congruence of L extends to K iff it is a $\vec{\varphi}$-congruence and every $\vec{\varphi}$-congruence of L has *exactly one* extension to K. If K is a $\vec{\varphi}$-congruence-preserving extension of L, then the congruence lattice of the partial algebra $L_{\vec{\varphi}}$ is isomorphic to the congruence lattice of the lattice K.

Let us call φ *algebraic* iff there is a unary algebraic function $\mathbf{p}(x)$ (that is, $\mathbf{p}(x)$ is obtained from a lattice polynomial by substituting all but one of the variables by elements of L) such that $x\varphi = \mathbf{p}(x)$, for all $x \in [a, b]$.

We prove the following result:

Theorem 1. *Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let φ be a homomorphism of $[a, b]$ onto $[c, d]$. Then L has a $\vec{\varphi}$-congruence-preserving convex extension into a bounded lattice K such that φ is algebraic in K. In particular, the congruence lattice of the partial algebra $L_{\vec{\varphi}}$ is isomorphic to the congruence lattice of the bounded lattice K. If L is finite, then K can be constructed as a finite lattice.*

So K realizes the "magic wand": $a \equiv b$ implies that $c \equiv d$.

Note that Theorem 1 of this paper implies Theorem 1 of Part I, which states that if L is a bounded lattice, $[a, b]$ and $[c, d]$ are intervals of L, and φ is an isomorphism of $[a, b]$ and $[c, d]$, then L has a bounded extension K such that a congruence Θ of L extends to K iff $x \equiv y$ (Θ) is equivalent to $x\varphi \equiv y\varphi$ (Θ), for all x, $y \in [a, b]$; in which case, Θ has exactly one extension to K. We obtain such an extension K by applying Theorem 1 of this paper to L to obtain a convex extension K_1 for φ, and then applying it to K_1, the intervals $[a, b]$ and $[c, d]$ (which remain intervals in K_1 because of the convexity of the extension) and φ^{-1} to obtain K. Observe that in this application, φ is an isomorphism, while in this paper, φ is only an onto homomorphism.

1.2. **Many surjective homomorphisms.** Let L be a bounded lattice, and for $i \in I$, let φ_i be a homomorphism of the interval $[a_i, b_i]$ onto the interval $[c_i, d_i]$. Let

$$\Phi = \{\, \varphi_i \mid i \in I \,\},$$

and let L_Φ denote the partial algebra obtained from L by adding the partial operations φ_i, for $i \in I$. Let us call a congruence Θ of L a Φ-*congruence* iff Θ satisfies the Substitution Property with respect to the partial unary operations φ_i, $i \in I$, that is, $x \equiv y$ (Θ) implies that $x\varphi_i \equiv y\varphi_i$ (Θ), for all x, $y \in [a_i, b_i]$ and $i \in I$. Thus, a congruence relation of L_Φ is the same as a Φ-congruence of L. We call K a Φ-*congruence-preserving extension* of L, if a congruence of L extends to K iff it is a Φ-congruence of L and every Φ-congruence of L has *exactly one* extension to K.

Theorem 2. *Let L be a bounded lattice, let Φ be given as above. Then the partial algebra L_Φ has a Φ-congruence-preserving convex extension into a lattice K such that all φ_i, $i \in I$, are algebraic in K. In particular, the congruence lattice of the partial algebra L_Φ is isomorphic to the congruence lattice of the lattice K.*

Theorem 2 of this paper easily implies Theorems 2 and 3 of Part I, with one important difference: In Theorem 2 of Part I, we obtain a *bounded* lattice K. Unfortunately, we do not know how to ensure the K be bounded in Theorem 2 of this paper; recall that we did not know how to obtain a bounded K in Theorem 3 of Part I. This shows why the two parts of this paper would be difficult to merge:

To obtain Theorem 2 of Part I, we would have to present the construction of Part I; the construction in this paper would be of no help.

1.3. **Outline.** The proofs are basically the same as in Part I, except that the lattice B of Part I is radically different from the lattice B of this paper. This new B is constructed using the "2/3-Boolean triple construction" described in Section 2. We prove Theorem 1 in Section 3; we just provide the arguments that are necessary to change the proof of Part I. Theorem 2 is proved in Section 4. Finally, in Section 5, we discuss some applications and a few open problem.

We use the standard notation, as in [2] and [6].

1.4. **Acknowledgment.** Again, just as in Part I, we would like to thank the referee for an unusually perceptive report, which resulted in a much improved paper.

2. The 2/3-Boolean triple construction

2.1. **The Boolean triple construction.** G. Grätzer and F. Wehrung [7], for a lattice P, introduced Boolean triples: the element $\langle x, y, z\rangle \in P^3$ is called a *Boolean triple* iff

$$\begin{aligned} x &= (x \vee y) \wedge (x \vee z),\\ y &= (y \vee x) \wedge (y \vee z),\\ z &= (z \vee x) \wedge (z \vee y). \end{aligned}$$

They proved that $M_3\langle P\rangle$, the set of all Boolean triples partially ordered componentwise, is a lattice, in fact, a congruence-preserving extension of P.

2.2. **The $N_6\langle P\rangle$ construction.** Let $N_6 = \{o, p, q_1, q_2, r, i\}$ denote the six-element lattice depicted in Figure 1, with o the zero, i the unit element, p, q_1, q_2 the atoms, satisfying the relations $q_1 \vee q_2 = r$, $p \wedge q_1 = p \wedge q_2 = p \wedge r = o$, and $p \vee q_1 = p \vee q_2 = p \vee r = i$.

In this paper, for a bounded lattice P, we introduce 2/3-Boolean triples: the element $\langle x, y, z\rangle \in P^3$ is called a 2/3-*Boolean triple* iff

$$\begin{aligned} y &= (y \vee x) \wedge (y \vee z),\\ z &= (z \vee x) \wedge (z \vee y). \end{aligned}$$

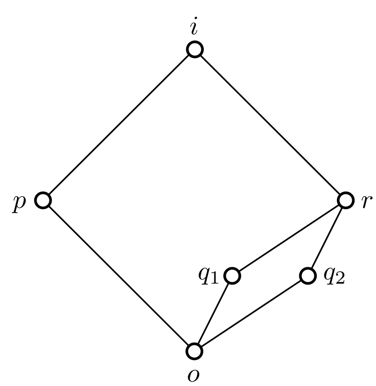

Figure 1. The lattice N_6.

We retain two out of three conditions that define Boolean triples, hence the name. The notation $N_6\langle P\rangle$ follows the notational convention of G. Grätzer and M. Greenberg [3] and G. Grätzer and F. Wehrung [8]; the role of N_6 in this construction is clarified by Corollary 2 and Lemma 3.

We denote by $N_6\langle P\rangle$ the set of all 2/3-Boolean triples partially ordered componentwise. In this section, we prove that $N_6\langle P\rangle$ is a lattice and describe the congruences of this lattice.

Lemma 1. *$N_6\langle P\rangle$ is a closure system in P^3; let $\overline{\langle x,y,z\rangle}$ denote the closure of $\langle x,y,z\rangle \in P^3$ and call it the* 2/3-Boolean closure *of $\langle x,y,z\rangle \in P^3$. Then*

$$\overline{\langle x,y,z\rangle} = \langle x, (y \vee x) \wedge (y \vee z), (z \vee x) \wedge (z \vee y)\rangle.$$

Proof. In this proof, let $\overline{y} = (y \vee x) \wedge (y \vee z)$ and $\overline{z} = (z \vee x) \wedge (z \vee y)$. We have to verify that $\overline{\langle x,y,z\rangle} = \langle x, \overline{y}, \overline{z}\rangle$ is the closure of $\langle x,y,z\rangle$.

The triple $\langle x, \overline{y}, \overline{z}\rangle$ is 2/3-Boolean closed. Indeed, $y \leq \overline{y}$, so $x \vee y = x \vee \overline{y}$. Also, $z \leq \overline{z}$, so $\overline{y} \vee \overline{z} = y \vee z$. Therefore,

$$(\overline{y} \vee x) \wedge (\overline{y} \vee \overline{z}) = \overline{y},$$

verifying the first half of the definition of 2/3-Boolean triples; the second half is proved similarly.

So $\langle x,y,z\rangle \leq \langle x,\overline{y},\overline{z}\rangle \in N_6\langle P\rangle$. To prove that $N_6\langle P\rangle$ is a closure system in P^3 and that $\langle x,\overline{y},\overline{z}\rangle$ is the closure of $\langle x,y,z\rangle$, it suffices to verify that if $\langle x_1,y_1,z_1\rangle \in N_6\langle P\rangle$ and $\langle x,y,z\rangle \leq \langle x_1,y_1,z_1\rangle$, then $\langle x,\overline{y},\overline{z}\rangle \leq \langle x_1,y_1,z_1\rangle$, which is obvious. ☐

Corollary 2. *$N_6\langle P\rangle$ is a lattice. Meet is componentwise and join is the closure of the componentwise join. Moreover, $N_6\langle P\rangle$ has a spanning N_6 (see Figure 2):*

$$\{o = \langle 0,0,0\rangle,\ p = \langle 1,0,0\rangle,\ q_1 = \langle 0,1,0\rangle,\ q_2 = \langle 0,0,1\rangle,\ r = \langle 0,1,1\rangle,\ i = \langle 1,1,1\rangle\}.$$

Lemma 3.

(i) *The interval $[o,p]$ of $N_6\langle P\rangle$ is isomorphic to P under the isomorphism*

$$x_p = \langle x,0,0\rangle \mapsto x, \quad x \in P.$$

(ii) *The interval $[o,q_1]$ of $N_6\langle P\rangle$ is isomorphic to P under the isomorphism*

$$x_{q_1} = \langle 0,x,0\rangle \mapsto x, \quad x \in P.$$

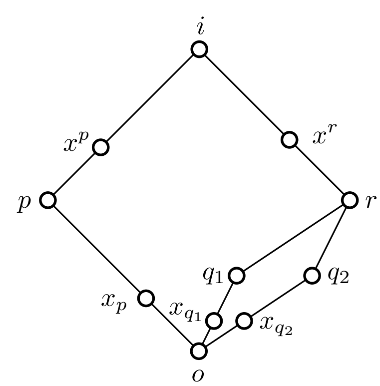

FIGURE 2. Illustrating $N_6\langle P\rangle$.

(iii) *The interval* $[o, q_2]$ *of* $N_6\langle P\rangle$ *is isomorphic to* P *under the isomorphism*

$$x_{q_2} = \langle 0, 0, x\rangle \mapsto x, \quad x \in P.$$

(iv) *The interval* $[p, i]$ *of* $N_6\langle P\rangle$ *is isomorphic to* P *under the isomorphism*

$$x^p = \langle 1, x, x\rangle \mapsto x, \quad x \in P.$$

(v) *The interval* $[o, r]$ *of* $N_6\langle P\rangle$ *is isomorphic to* P^2 *under the isomorphism*

$$\langle 0, x, y\rangle \mapsto \langle x, y\rangle, \quad x,\ y \in P.$$

(vi) *The interval* $[r, i]$ *of* $N_6\langle P\rangle$ *is isomorphic to* P *under the isomorphism*

$$x^r = \langle x, 1, 1\rangle \mapsto x, \quad x \in P.$$

Proof. By trivial computation. For instance, to prove (iv), observe that $\langle 1, x, y\rangle$ is closed iff $x = y$. □

For the five isomorphic copies of P in $N_6\langle P\rangle$, we use the notation:

$$\begin{aligned} P_p &= [o, p],\\ P_{q_1} &= [o, q_1],\\ P_{q_2} &= [o, q_2], \end{aligned}$$

with zero o and unit elements, p, q_1, q_2, respectively, and

$$\begin{aligned} P^p &= [p, i],\\ P^r &= [r, i]. \end{aligned}$$

with unit 1 and zero elements, p, r, respectively.

2.3. **The congruences of** $N_6\langle P\rangle$**.** We describe the congruence structure of $N_6\langle P\rangle$ based on the following decomposition of elements:

Lemma 4. *Every* $\alpha \in N_6\langle P\rangle$ *has a decomposition*

$$\alpha = (\alpha \wedge p) \vee (\alpha \wedge q_1) \vee (\alpha \wedge q_2),$$

where $\alpha \wedge p \in P_p$*,* $\alpha \wedge q_1 \in P_{q_1}$*, and* $\alpha \wedge q_2 \in P_{q_2}$*.*

Proof. Indeed, the componentwise join of the right side equals α. □

For a congruence Ψ of $N_6\langle P\rangle$, let Ψ_p denote the restriction of Ψ to P_p, same for Ψ_{q_1} and Ψ_{q_2}. Let $\hat{\Psi}_p$ denote Ψ_p regarded as a congruence of P; same for $\hat{\Psi}_{q_1}$ and $\hat{\Psi}_{q_2}$. Similarly, let Ψ^p and Ψ^r denote the restriction of Ψ to P^p and P^r, respectively, and let $\hat{\Psi}^p$ and $\hat{\Psi}^r$ denote the corresponding congruences of P. Then we obtain:

Lemma 5. *Let* α*,* $\alpha' \in N_6\langle P\rangle$ *and* $\Psi \in \operatorname{Con} N_6\langle P\rangle$*. Then*

$$\alpha \equiv \alpha' \quad (\Psi)$$

iff

$$\begin{aligned} \alpha \wedge p &\equiv \alpha' \wedge p \quad (\Psi_p),\\ \alpha \wedge q_1 &\equiv \alpha' \wedge q_1 \quad (\Psi_{q_1}),\\ \alpha \wedge q_2 &\equiv \alpha' \wedge q_2 \quad (\Psi_{q_2}). \end{aligned}$$

Proof. This is clear from Lemma 4. □

Lemma 6. *Let $\alpha = \langle x, y, z\rangle$, $\alpha' = \langle x', y', z'\rangle \in N_6\langle P\rangle$ and $\Psi \in \operatorname{Con} N_6\langle P\rangle$. Then*

$$\alpha \equiv \alpha' \quad (\Psi)$$

iff

$$\begin{aligned} x &\equiv x' \quad (\hat{\Psi}_p),\\ y &\equiv y' \quad (\hat{\Psi}_{q_1}),\\ z &\equiv z' \quad (\hat{\Psi}_{q_2}). \end{aligned}$$

Proof. This is clear from Lemma 5. □

Now we have the tools to describe the congruences. The next four lemmas provide the description.

Lemma 7. *Let $\Psi \in \operatorname{Con} N_6\langle P\rangle$. Then $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$ in P.*

Proof. Indeed, if $u \equiv u'$ $(\hat{\Psi}_{q_1})$, then $\langle 0, u, 0\rangle \equiv \langle 0, u', 0\rangle$ (Ψ_{q_1}), so $\langle 0, u, 0\rangle \equiv \langle 0, u', 0\rangle$ (Ψ). Therefore,

$$\begin{aligned} \langle 0, 0, u\rangle &= (\langle 0, u, 0\rangle \vee \langle 1, 0, 0\rangle) \wedge \langle 0, 0, 1\rangle\\ &\equiv (\langle 0, u', 0\rangle \vee \langle 1, 0, 0\rangle) \wedge \langle 0, 0, 1\rangle = \langle 0, 0, u'\rangle\,(\Psi). \end{aligned}$$

We conclude that $\langle 0, 0, u\rangle \equiv \langle 0, 0, u'\rangle$ (Ψ_{q_2}), that is, $u \equiv u'$ $(\hat{\Psi}_{q_2})$, proving that $\hat{\Psi}_{q_1} \leq \hat{\Psi}_{q_2}$. By symmetry, $\hat{\Psi}_{q_1} \geq \hat{\Psi}_{q_2}$, so $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$. □

Lemma 8. *For $\Psi \in \operatorname{Con} N_6\langle P\rangle$, the congruence inequality $\hat{\Psi}_p \leq \hat{\Psi}_{q_2}$ holds.*

Proof. Indeed, if $u \equiv u'$ $(\hat{\Psi}_p)$, then $\langle u, 0, 0\rangle \equiv \langle u', 0, 0\rangle$ (Ψ_p). Therefore,

$$\begin{aligned} \langle 0, 0, u\rangle = \langle u, 1, u\rangle \wedge \langle 0, 0, 1\rangle &= (\langle u, 0, 0\rangle \vee \langle 0, 1, 0\rangle) \wedge \langle 0, 0, 1\rangle\\ &\equiv (\langle u', 0, 0\rangle \vee \langle 0, 1, 0\rangle) \wedge \langle 0, 0, 1\rangle\\ &= \langle u', 1, u'\rangle \wedge \langle 0, 0, 1\rangle = \langle 0, 0, u'\rangle\,(\Psi), \end{aligned}$$

that is, $u \equiv u'$ $(\hat{\Psi}_{q_2})$, proving that $\hat{\Psi}_p \leq \hat{\Psi}_{q_2}$. □

Lemma 9. *Let $\Theta \leq \Phi \in \operatorname{Con} P$. Then there is a unique $\Psi \in \operatorname{Con} N_6\langle P\rangle$, such that $\hat{\Psi}_p = \Theta$ and $\hat{\Psi}_{q_1} = \hat{\Psi}_{q_2} = \Phi$.*

Proof. The uniqueness follows from the previous lemmas. To prove the existence, for $\Theta \leq \Phi \in \operatorname{Con} P$, define a congruence Ψ on $N_6\langle P\rangle$ by

$$\langle x, y, z\rangle \equiv \langle x', y', z'\rangle\,(\Psi)$$

iff

$$\begin{aligned} x &\equiv x'\,(\Theta),\\ y &\equiv y'\,(\Phi),\\ z &\equiv z'\,(\Phi). \end{aligned}$$

It is obvious that Ψ is an equivalence relation and it satisfies the Substitution Property for meet. To verify the Substitution Property for join, let $\langle x, y, z\rangle \equiv \langle x', y', z'\rangle \ (\Psi)$ and let $\langle u, v, w\rangle \in N_6\langle P\rangle$. Then

$$\begin{aligned}&\langle x, y, z\rangle \vee \langle u, v, w\rangle = \overline{\langle x \vee u, y \vee v, z \vee w\rangle}\\ &\quad = \langle x \vee u, (x \vee y \vee u \vee v) \wedge (y \vee z \vee v \vee w), (x \vee z \vee u \vee w) \wedge (y \vee z \vee v \vee w)\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}&\langle x', y', z'\rangle \vee \langle u, v, w\rangle\\ &= \langle x' \vee u, (x' \vee y' \vee u \vee v) \wedge (y' \vee z' \vee v \vee w), (x' \vee z' \vee u \vee w) \wedge (y' \vee z' \vee v \vee w)\rangle.\end{aligned}$$

Since $x \equiv x' \ (\Theta)$, we also have

$$x \vee u \equiv x' \vee u \quad (\Theta).$$

From $x \equiv x' \ (\Theta)$ and $\Theta \leq \Phi$, it follows that $x \equiv x' \ (\Phi)$. Also, $y \equiv y' \ (\Phi)$, so $x \vee y \equiv x' \vee y' \ (\Phi)$. Therefore, $x \vee y \vee u \vee v \equiv x' \vee y' \vee u \vee v \ (\Phi)$. Similarly (or even simpler), $y \vee z \vee v \vee w \equiv y' \vee z' \vee v \vee w \ (\Phi)$. Meeting the last two congruences, we obtain that

$$(x \vee y \vee u \vee v) \wedge (y \vee z \vee v \vee w) \equiv (x' \vee y' \vee u \vee v) \wedge (y' \vee z' \vee v \vee w) \, (\Phi).$$

Similarly,

$$(x \vee z \vee u \vee w) \wedge (y \vee z \vee v \vee w) \equiv (x' \vee z' \vee u \vee w) \wedge (y' \vee z' \vee v \vee w) \, (\Phi).$$

The last three displayed equations verify that

$$\langle x, y, z\rangle \vee \langle u, v, w\rangle \equiv \langle x', y', z'\rangle \vee \langle u, v, w\rangle \, (\Phi).$$ ☐

Now note that for x, $y \in P$ and congruence Ψ of P,

$$x_p \equiv y_p \quad (\Psi) \qquad \text{iff} \qquad x^r \equiv y^r \quad (\Psi)$$

and

$$x^p \equiv y^p \quad (\Psi) \qquad \text{iff} \qquad x_{q_1} \equiv y_{q_1} \quad (\Psi).$$

It follows that $\hat{\Psi}^p = \hat{\Psi}_{q_1} = \hat{\Psi}_{q_2}$ and $\hat{\Psi}^r = \hat{\Psi}_p$, so Lemmas 7–9 can be restated as follows:

Corollary 10. *There is a one-to-one correspondence between the congruences of $N_6\langle P\rangle$ and pairs of congruences $\Theta \leq \Phi$ of $\operatorname{Con} P$, defined by*

$$\Psi \mapsto \langle \hat{\Psi}^r, \hat{\Psi}^p\rangle.$$

Lemma 9 and Corollary 10 can be proved using lattice tensor products introduced in G. Grätzer and F. Wehrung [8]. The lattice $N_6\langle P\rangle$ is isomorphic to the lattice tensor product of N_6 and P; an isomorphism is given by

$$\langle x, y, z\rangle \mapsto (p \boxtimes x) \vee (q_1 \boxtimes x) \vee (q_2 \boxtimes x).$$

The results in [8] can be used to verify these lemmas.

In the special case when one of the factors is finite, G. Grätzer and M. Greenberg [3] provide a much more elementary approach to lattice tensor products by providing a coordinatization. Our definition of $N_6\langle P\rangle$ is based on this coordinatization, especially, on Section 3 of [3].

By the main result of G. Grätzer and M. Greenberg [4] (using the notation of [4]),

$$\begin{aligned} \mathrm{Con}\, N_6\langle P\rangle &\cong (\mathrm{Con}\, N_6)\langle \mathrm{Con}\, P\rangle \\ &\cong C_3\langle \mathrm{Con}\, P\rangle \\ &\cong \{\, \langle \Theta, \Phi\rangle \mid \Theta \leq \Phi \text{ in } \mathrm{Con}\, P \,\}, \end{aligned}$$

where C_3 is the three-element chain. But there is more in [4]; the isomorphism is explicitly exhibited. Let $\mathcal{F}\colon \mathrm{Con}\, N_6\langle P\rangle \to (\mathrm{Con}\, N_6)\langle \mathrm{Con}\, P\rangle$ be the isomorphism. Then by definition (see equation (11) in [4]), $\mathcal{F}(\Psi)(\Theta_{N_6}(u, v)) = \Psi_{uv}$. Therefore, $\mathcal{F}\colon \Psi \mapsto \langle \Psi_{0p}, \Psi_{0q_1}\rangle$. In other words, each congruence of $N_6\langle P\rangle$ is obtained by taking two congruences $\Theta \leq \Phi$ of P, imposing Θ on the interval $[0, p] \subseteq N_6\langle P\rangle$, and Φ on the interval $[0, q_1] \subseteq N_6\langle P\rangle$. This verifies again Lemma 9 and Corollary 10.

2.4. **An algebraic function on $N_6\langle P\rangle$.** The inequality $\hat{\Psi}^r \leq \hat{\Psi}^p$ can be established in a stronger form by exhibiting an algebraic function $\mathbf{r}(x)$ on $N_6\langle P\rangle$ such that $\mathbf{r}(u^r) = u^p$, for $u \in P$. We now proceed to exhibit $\mathbf{r}(x)$, which we shall need in Section 3.

Lemma 11. *There is an algebraic function $\mathbf{r}(x)$ on $N_6\langle P\rangle$ such that $\mathbf{r}(u^r) = u^p$, for $u \in P$.*

Proof. Define

$$\mathbf{r}(x) = (((x \wedge p) \vee q_1) \wedge q_2) \vee p.$$

Indeed, if $u \in P$, then $u^r = \langle u, 1, 1\rangle$ and so

$$\begin{aligned} \mathbf{r}(u^r) &= \mathbf{r}(\langle u, 1, 1\rangle) \\ &= (((\langle u, 1, 1\rangle \wedge \langle 1, 0, 0\rangle) \vee \langle 0, 1, 0\rangle) \wedge \langle 0, 0, 1\rangle) \vee \langle 1, 0, 0\rangle \\ &= ((\langle u, 0, 0\rangle \vee \langle 0, 1, 0\rangle) \wedge \langle 0, 0, 1\rangle) \vee \langle 1, 0, 0\rangle \\ &= (\overline{\langle u, 1, 0\rangle} \wedge \langle 0, 0, 1\rangle) \vee \langle 1, 0, 0\rangle \\ &= (\langle u, 1, u\rangle \wedge \langle 0, 0, 1\rangle) \vee \langle 1, 0, 0\rangle \\ &= \langle 0, 0, u\rangle \vee \langle 1, 0, 0\rangle \\ &= \overline{\langle 1, 0, u\rangle} \\ &= \langle 1, u, u\rangle = u^p, \end{aligned}$$

as required. ☐

2.5. **A quotient of $N_6\langle P\rangle$.** Actually, to prove Theorem 1, we need not the lattice $N_6\langle P\rangle$, but a quotient thereof, which we now proceed to construct.

Let P and Q be bounded lattices of more than one element, and let

$$\varphi\colon P \to Q$$

be a homomorphism of P *onto* Q. We denote by $\ker\varphi$ the *kernel* of this homomorphism, $\ker\varphi \in \mathrm{Con}\, P$.

By Corollary 10, there is a unique congruence Ψ of $N_6\langle P\rangle$ corresponding to the congruence pair $\omega \leq \ker\varphi$ of P. Define

$$B = N_6\langle P\rangle/\Psi.$$

It is useful to note that B can be represented as

$$\{\, \langle x, y, z\rangle \in P \times Q \times Q \mid y = (y \vee x\varphi) \wedge (y \vee z) \text{ and } z = (z \vee x\varphi) \wedge (z \vee y) \,\}.$$

By the Second Isomorphism Theorem (see, e.g., [1]), there is a one-to-one correspondence between the congruences of B and congruence pairs $\Theta \leq \Phi$ of P satisfying $\ker\varphi \leq \Phi$. For $x \in N_6\langle P\rangle$, let $\overline{x}$ denote the congruence class $[x]\Psi$.

Using this notation, utilizing the results of this section, we state some important properties of the lattice B:

Lemma 12. *Let P and Q be bounded lattices and let $\varphi\colon P \to Q$ be a homomorphism of P onto Q. Then there is a lattice B, with the following properties:*

- (i) *B has a spanning sublattice $\overline{o}$, $\overline{p}$, $\overline{q_1}$, $\overline{q_2}$, $\overline{r}$, $\overline{i}$ isomorphic to N_6.*
- (ii) *The interval $[\overline{r}, \overline{i}]$ is isomorphic to P under the map $x \mapsto \overline{x^r}$, $x \in P$.*
- (iii) *The interval $[\overline{p}, \overline{i}]$ is isomorphic to Q under the map $y \mapsto \overline{y^p}$, $y \in Q$, where $x \in P$ with $x\varphi = y$.*
- (iv) *The congruences Σ of B are in one-to-one correspondence with pairs of congruences $\langle\Theta, \Phi\rangle$, where Θ is a congruence of P and Φ is a congruence of Q satisfying $\Theta \leq \Phi\varphi^{-1}$, where up to isomorphism, Σ restricted to $[\overline{r}, \overline{i}]$ is Θ and Σ restricted to $[\overline{p}, \overline{i}]$ is Φ.*
- (v) *There is an algebraic function $\mathbf{r}(x)$ such that $\mathbf{r}(\overline{u^r}) = \overline{x^p}$, for $u \in P$, where $x \in P$ with $x\varphi = u$.*

3. Proof of Theorem 1

We proceed with the proof of Theorem 1 as in Part I. We can assume, without loss of generality, that $[a, b]$ and $[c, d]$ are nontrivial intervals, that is, $a < b$ and $c < d$.

3.1. **Constructing K.** We take the four building blocks: $A = M_3\langle L\rangle$, $L_{a,b}$, B, and $L_{c,d}$, except that now B is not $M_3\langle[a, b]\rangle$ but the lattice $B = N_6\langle[a, b]\rangle/\Psi$ as described in Section 2.5 (especially in Lemma 12), constructed from $P = [a, b]$, $Q = [c, d]$, and the homomorphism φ from $[a, b]$ onto $[c, d]$ given in the assumptions of Theorem 1.

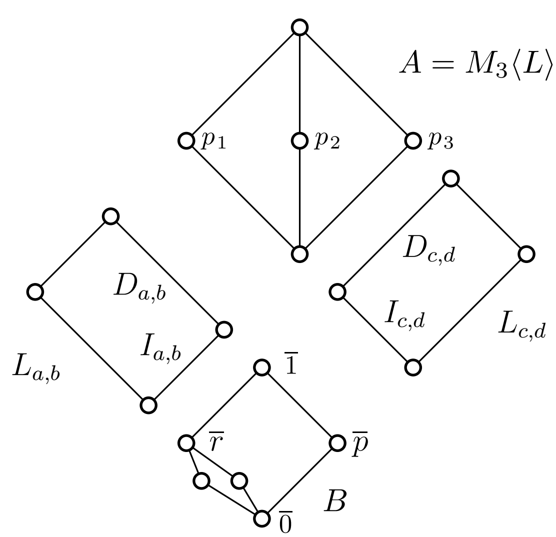

Figure 3. The four building blocks of K.

We do three gluings.

First gluing. In B, we use the dual ideal

$$[\overline{r}) = \{\, \overline{\langle x,1,1\rangle} \mid a \leq x \leq b \,\}$$

(which is isomorphic to $[a,b]$ since Ψ on $[r,1]$ is ω), while in $L_{a,b}$ we utilize the ideal

$$I_{a,b} = \{\, \langle 0,x,0\rangle \mid a \leq x \leq b \,\}$$

(which is obviously isomorphic to $[a,b]$), and we consider the natural isomorphism

$$\varphi_1\colon \overline{\langle x,1,1\rangle_B} \mapsto \langle 0,x,0\rangle_{L_{a,b}}, \quad x \in [a,b],$$

between the dual ideal $[\overline{r})$ of B and the ideal $I_{a,b}$ of $L_{a,b}$ to glue B and $L_{a,b}$ together to obtain the lattice U. (As in Part I, we use the following notation: to indicate whether a triple belongs to $A = M_3\langle L\rangle$, $L_{a,b}$, B, or $L_{c,d}$, we subscript the triple with A, $L_{a,b}$, B, or $L_{c,d}$, respectively.)

Second gluing. We glue $L_{c,d}$ and A over the dual ideal

$$D_{c,d} = \{\, \langle x,c,x \wedge c\rangle \mid x \in L \,\}$$

of $L_{c,d}$ and the ideal

$$(p_3] = \{\, \langle 0,0,x\rangle \mid x \in L \,\}$$

of A, with respect to the natural isomorphism

$$\varphi_2\colon \langle x,c,x \wedge c\rangle_{L_{c,d}} \mapsto \langle 0,0,x\rangle_A, \quad x \in L,$$

to obtain the lattice V.

Final gluing. In U, we define the dual ideal

$$D = [\overline{p},\overline{1}] \cup D_{a,b},$$

which is the union of $[\overline{p},\overline{1}]$ and $D_{a,b}$, with the unit of $[\overline{p},\overline{1}]$ identified with the zero of $D_{a,b}$.

In V, we define the ideal

$$I = I_{c,d} \cup [0_A, p_1],$$

which is the union of $I_{c,d}$ and $[0_A, p_1]$, with the unit of $I_{c,d}$ identified with the zero of $[0_A, p_1]$.

Next we set up an isomorphism $\psi\colon D \to I$. Since

$$[\overline{p},\overline{1}] = \{\, \overline{\langle 1,x,x\rangle_B} \mid a \leq x \leq b \,\}$$

and

$$I_{c,d} = \{\, \langle 0,x,0\rangle_{L_{c,d}} \mid c \leq x \leq d \,\},$$

we define ψ on $[\overline{p},\overline{1}]$ by

$$\psi\colon \overline{\langle 1,x,x\rangle_B} \mapsto \langle 0,x\varphi,0\rangle_{L_{c,d}},$$

where $\varphi\colon [a,b] \to [c,d]$ is the isomorphism given in Theorem 1. We define ψ on $D_{a,b}$ by

$$\psi\colon \langle x,b,x \wedge b\rangle_{L_{a,b}} \mapsto \langle x,0,0\rangle_A, \quad x \in L.$$

It is clear that $\psi\colon D \to I$ is well-defined and it is an isomorphism.

Finally, we construct the lattice K of Theorem 1 by gluing U over I with V over D with respect to the isomorphism $\psi\colon D \to I$.

The map $x \mapsto \langle x,0,0\rangle_A$ is a natural isomorphism between L and the principal ideal $(p_1]$ of A; this gives us a convex embedding of L into A. We identify L with its image, and regard L as a convex sublattice of A and therefore of K. So K is a convex extension of L. We have completed the construction of the bounded lattice K of Theorem 1.

3.2. **Congruences of K.** The proof in Part I heavily depended on the fact that we glued over ideals and dual ideals of which the building components were congruence-preserving extensions. This is no longer the case; however, a modification of Lemma 11 of Part I comes to the rescue.

A congruence Ω of K can be described by four congruences,

Ω_A, the restriction of Ω to A,
$\Omega_{a,b}$, the restriction of Ω to $L_{a,b}$,
$\Omega_{c,d}$, the restriction of Ω to $L_{c,d}$,
Ω_B, the restriction of Ω to B.

These congruences satisfy a number of conditions:

(i) $\langle 0,x,0\rangle_{L_{a,b}} \equiv \langle 0,y,0\rangle_{L_{a,b}}$ $(\Omega_{a,b})$ iff $\overline{\langle x,1,1\rangle_B} \equiv \overline{\langle y,1,1\rangle_B}$ (Ω_B), for $x,\ y \in [a,b]$.
(ii) $\langle 0,0,x\rangle_A \equiv \langle 0,0,y\rangle_A$ (Ω_A) iff $\langle x,c,x\wedge c\rangle_{L_{c,d}} \equiv \langle y,c,y\wedge c\rangle_{L_{c,d}}$ $(\Omega_{c,d})$, for $x,\ y \in L$.
(iii) $\langle x,0,0\rangle_A \equiv \langle y,0,0\rangle_A$ (Ω_A) iff $\langle x,a,x\wedge a\rangle_{L_{a,b}} \equiv \langle y,a,y\wedge a\rangle_{L_{a,b}}$ $(\Omega_{a,b})$, for $x,\ y \in L$.
(iv) $\langle 0,x\varphi,0\rangle_{L_{c,d}} \equiv \langle 0,y\varphi,0\rangle_{L_{c,d}}$ $(\Omega_{c,d})$ iff $\overline{\langle 1,x,x\rangle_B} \equiv \overline{\langle 1,y,y\rangle_B}$ (Ω_B), for $x,\ y \in [a,b]$.

Conversely, if we are given congruences Ω_A on A, $\Omega_{a,b}$ on $L_{a,b}$, $\Omega_{c,d}$ on $L_{c,d}$, Ω_B on B, then by (i), we can define a congruence Ω_U on U. By (ii), we can define a congruence Ω_V on V. By (iii) and (iv), we can define a congruence Ω_K on K.

Now it is clear that if we start with a congruence Σ of L, then we can define the congruences Σ_A on A, $\Sigma_{a,b}$ on $L_{a,b}$, $\Sigma_{c,d}$ on $L_{c,d}$ componentwise, and Σ_B on B as in Section 2.5. Conditions (i)–(iii) trivially hold (since Σ_A, $\Sigma_{a,b}$, and $\Sigma_{c,d}$ are defined componentwise). Finally, (iv) holds if Σ is a $\vec{\varphi}$-congruence. So every $\vec{\varphi}$-congruence of L has an extension to K.

Let Σ be a congruence of L that extends to K. Since A is a congruence-preserving convex extension of $L = [0_A, p_1]$, further, $L_{a,b}$ is a congruence-preserving extension of $D_{a,b}$, and $L_{c,d}$ is a congruence-preserving extension of $D_{c,d}$, the congruence Σ uniquely extends to A as Σ_A, to $L_{a,b}$ as $\Sigma_{a,b}$, and to $L_{c,d}$ as $\Sigma_{c,d}$. Therefore, Σ uniquely extends to the intervals $[\overline{r},\overline{1}]$ and $[\overline{p},\overline{1}]$ of B, and so by Lemma 12 to B. We conclude that if a congruence Σ of L extends to K, then it extends uniquely.

To complete the proof, we prove that φ is algebraic. Define

$$\begin{aligned}\mathbf{p}(x) = (((((((((x \wedge \langle a,a,b\rangle_{L_{a,b}}) \vee \langle b,a,a\rangle_{L_{a,b}}) \wedge p) \vee q_1) \wedge q_2)\\ \vee \langle d,c,c\rangle_{L_{c,d}}) \wedge \langle c,c,d\rangle_{L_{c,d}}) \vee p_2) \wedge p_1.\end{aligned}$$

By Lemma 11, $\mathbf{p}(x)$ behaves properly in B, while outside of B, $\mathbf{p}(x)$ is the same algebraic function as in Part I.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We can assume, for $i \in I$, without loss of generality, that $[a_i, b_i]$ and $[c_i, d_i]$ are nontrivial intervals, that is, $a_i < b_i$ and $c_i < d_i$. We can assume that $|I| > 2$ by repeating an interval and a homomorphism, if necessary.

As in Section 3.1—*mutatis mutandis*—we take the building blocks: $A = M_I\langle L\rangle$ (where M_I is the length two modular lattice with atoms p_i, for $i \in I$), L_{a_i,b_i}, B_i, and L_{c_i,d_i}, for $i \in I$. Let B be the dual discrete direct product of the B_i, $i \in I$. Note that B does not have a zero in general. Now we proceed as in Section 7 of Part I: For $i \in I$, we glue $L(a_i, b_i)$ to A (using the ideal $[0, p_i]$ of A) and at the same time glue it to B_i—and therefore, to B—using the dual ideal $[\overline{r}_i)$ of B_i (therefore, of B).

We can verify that L has the required properties as in Part I, except that now we rely on Lemma 12 when we utilize the properties of B. This completes the proof of Theorem 2.

Note that Theorem 2 generalizes both Theorems 2 and 3 of Part I. However, in spirit, it is closer to Theorem 3 of Part I. Indeed, in Theorem 2 of Part I, we consider a family $[a_i, b_i]$, $i < \alpha$, of intervals of L, and the isomorphisms

$$\varphi_{i,j}\colon [a_i, b_i] \to [a_j, b_j], \quad \text{for } i,\ j < \alpha.$$

The isomorphisms are assumed to satisfy the natural "associativity" conditions, so they are interdependent.

In Theorem 3 of Part I, we are given a doubly indexed family $[a_i^m, b_i^m]$, $i < \alpha$, $m < \mu$, of intervals of L, and the isomorphisms

$$\varphi_{i,j}^m\colon [a_i^m, b_i^m] \to [a_j^m, b_j^m], \quad \text{for } i,\ j < \alpha,\ m < \mu.$$

The isomorphisms $\varphi_{i,j}^m$, for m fixed, are assumed to satisfy the natural "associativity" conditions, however, the isomorphisms $\varphi_{i,j}^m$ and $\varphi_{i,j}^n$ do not interact for $m \neq n$, with m, $n < \mu$. Similarly, in Theorem 2 of this paper, the homomorphisms φ_i, for $i \in I$, do not interact for $i \neq j$, with i, $j \in I$. In both cases, we attain this "independence" by forming the dual discrete direct product of the B_i, which results in a lattice without a zero element.

5. Discussion

5.1. **Convex sublattices.** Theorem 2 allows us to generalize Theorem 1 from intervals to convex sublattices. Let L be a bounded lattice, let U and V be convex sublattices of L, and let φ be a homomorphism from U *onto* V. We introduce a $\overrightarrow{\varphi}$*-congruence*, the partial algebra $L_{\overrightarrow{\varphi}}$, and a $\overrightarrow{\varphi}$*-congruence-preserving extension* as in Section 1.1, *mutatis mutandis*.

Let L be a lattice, let A, $B \subseteq L$, and let $\varphi\colon A \to B$ be a map. We call the map φ *locally algebraic* iff for every $[u, v] \subseteq A$, there is a unary algebraic function $\mathbf{p}(x)$ such that $x\varphi = \mathbf{p}(x)$, for all $x \in [u, v]$. For an interval A, locally algebraic is the same as algebraic.

Here is the generalization of Theorem 1:

Theorem 1′. *Let L be a bounded lattice, let U and V be convex sublattices of L, and let φ be a homomorphism from U* onto *V. Then L has a $\overrightarrow{\varphi}$-congruence-preserving convex extension into a lattice K such that φ is locally algebraic in K. In particular, the congruence lattice of the partial algebra $L_{\overrightarrow{\varphi}}$ is isomorphic to the congruence lattice of the lattice K.*

Note that K is no longer claimed to be bounded because we obtain it from Theorem 2.

Proof. Let $[a_i, b_i]$, $i \in I$, be the family of all subintervals of U. Let $[c_i, d_i]$, $i \in I$, be the family of all corresponding subintervals of V, that is, $c_i = a_i\varphi$ and $d_i = b_i\varphi$, for all $i \in I$, and define $\varphi_i \colon [a_i, b_i] \to [c_i, d_i]$ as the restriction of φ to $[a_i, b_i]$, for $i \in I$. Now we get Theorem 1′ by a straightforward application of Theorem 2. □

Of course, we can similarly generalize Theorem 2.

5.2. **Fully invariant congruences.** As usual, let us call a congruence Θ *fully invariant* iff $a \equiv b\ (\Theta)$ implies that $a\alpha \equiv b\alpha\ (\Theta)$, for any automorphism α.

For a lattice L, let $\operatorname{Con}_{\mathrm{inv}} L$ denote the lattice of fully invariant congruences of L, and let $\operatorname{Aut} L$ denote the set (group) of automorphisms of L.

For a bounded lattice L, we can apply Theorem 2 to $I = \operatorname{Aut} L$; for $\alpha \in \operatorname{Aut} L$, let $[a_\alpha, b_\alpha] = [c_\alpha, d_\alpha] = [0, 1]$, and let $\varphi_\alpha = \alpha$.

Theorem 3. *Let L be a bounded lattice. Then L has a convex extension into a lattice K such that a congruence of L extends to K iff it is fully invariant and a fully invariant congruence of L extends uniquely to K. In particular,* $\operatorname{Con}_{\mathrm{inv}} L$ *is isomorphic to* $\operatorname{Con} K$.

5.3. **The 1/3-Boolean triple construction.** The reader may ask what is a 1/3-Boolean triple construction? For a lattice P, let us call the element $\langle x, y, z\rangle \in P^3$ a 1/3-*Boolean triple* iff

$$z = (z \vee x) \wedge (z \vee y).$$

Then instead of N_6 of Figure 1, we now get the lattice N_7 of Figure 4 (the dual of the seven-element semimodular but not modular lattice), and the 1/3-Boolean triples form a lattice isomorphic to $N_7\langle P\rangle$, using the notation of G. Grätzer and M. Greenberg [3], which, in turn, is isomorphic to $N_7 \boxtimes P$, using the notation of G. Grätzer and F. Wehrung [8].

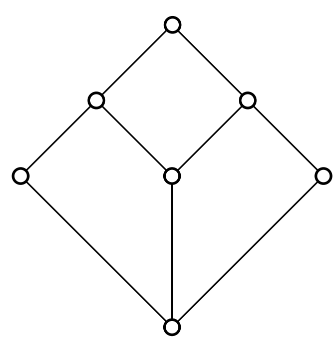

FIGURE 4. The lattice N_7.

5.4. **Finite lattices.** As a very simple application, we verify a result of R. P. Dilworth (first published in G. Grätzer and E. T. Schmidt [5]):

Theorem 4. *Every finite distributive lattice can be represented as the congruence lattice of a finite lattice.*

Proof. Let D be a finite distributive lattice, and let $P = \mathrm{J}(D)$ be the poset of join-irreducible elements of D. Let L be the Boolean lattice with atoms a_i, $i \in P$. For $i > j$ in P, define $\varphi_{i,j}$, the only homomorphism of $[0, a_i]$ onto $[0, a_j]$. Apply Theorem 1 to these homomorphisms one at a time (or collectively apply Theorem 2) to construct a lattice K whose congruence lattice is isomorphic to D. □

Note that the results of Part I are not sufficient to provide this application: They only make intervals congruence equivalent. Here we want the intervals "congruence ordered".

5.5. **Congruence Lattice Problem.** As discussed in Part I of this paper in Section 9, the fundamental unsolved problem in this field is the Congruence Lattice Problem: *Can every distributive algebraic lattice be represented as the congruence lattice of a lattice?*

The best method to attack this problem was found by E. T. Schmidt [9]. A recent result of F. Wehrung [11] shows the limitations of this method. See J. Tůma and F. Wehrung [10] for a review of related recent results.

It is natural to ask whether the method of E. T. Schmidt [9] can be combined with the results of this paper to solve the Congruence Lattice Problem. It is pointed out in J. Tůma and F. Wehrung [10] that this is not the case.

However, we can ask the following question.

Problem 1. Can every distributive algebraic lattice be represented as the congruence lattice of a partial algebra of the form $F_0(\mathfrak{m})_\Phi$?

($F_0(\mathfrak{m})_\Phi$ denotes the free lattice with zero on $\mathfrak{m}$ generators.)

A positive answer to this question would partially answer Problem 2 of J. Tůma and F. Wehrung [10].

5.6. **Bounds.** Theorems 1 and 2 are proved in this paper only for bounded lattices. It is easy to see, however, that with minor technical changes we can prove them for arbitrary lattices L with zero. The lattice K we then obtain for Theorem 1 will have a zero but not necessarily a unit. The lattice K we then obtain for Theorem 2 may have neither zero nor unit.

In Theorem 1, we start with a bounded lattice L, and obtain a bounded lattice K. However, the construction does not preserve the bounds.

Problem 2. Can we strengthen Theorem 1 to obtain a $\{0,1\}$-preserving (convex) extension?

In Theorem 2 the lattice constructed may not have a zero.

Problem 3. Can we strengthen Theorem 2 to obtain a bounded (convex) extension? Even stronger: a bounded (convex) $\{0,1\}$-preserving extension?

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