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CONGRUENCE LATTICES OF UNIFORM LATTICES

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Abstract. A lattice L is *uniform*, if for any congruence Θ of L, any two congruence classes A and B of Θ are of the same size, that is, $|A| = |B|$ holds.

A classical result of R. P. Dilworth represents a finite distributive lattice D as the congruence lattice of a finite lattice L. We show that this L can be constructed as a finite uniform lattice.

1. Introduction

Why aren't lattices more like groups and rings? One can give, of course, many reasons, but one of them surely is the behavior of congruences. In groups and rings, there is a "neutral element", and the congruences are determined by the congruence classes containing this element. In a typical lattice, there is no such element. In groups and rings, any two congruence classes are of the same size; this is of course false in most lattices.

The congruence lattice of a finite lattice L is characterized by a classical result of R. P. Dilworth as a finite distributive lattice D. Many papers were published improving this result by representing a finite distributive lattice D as the congruence lattice of a finite lattice L with additional properties. These results are discussed–as of 1998–in detail in Section 1.7 of Appendix A (by the first author) and in Section 1 of Appendix C (by the first and second authors) of [1].

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A congruence Θ of a lattice L is *uniform*, if any two congruence classes A and B of Θ are of the same size, that is, $|A| = |B|$ holds. A lattice L is *uniform*, if all of its congruences are uniform.

We prove the following result:

Theorem 1.1. *Every finite distributive lattice D can be represented as the congruence lattice of a finite uniform lattice L.*

In G. Grätzer and E. T. Schmidt [3], we proved that every finite lattice has a congruence-preserving extension to a finite, sectionally complemented lattice, in particular, every finite distributive lattice D can be represented as the congruence lattice of a finite, sectionally complemented lattice L. Sectionally complemented lattices form another class of lattices that is more like groups and rings. Are these two classes of lattices the same?

No, they are not. The lattice N_6 (which you obtain from the five-element modular nondistributive lattice $N_5 = \{o, a, b, c, i\}$ with $a < b$ by adjoining a relative complement t of a in $[o, b]$; see, for instance, Figure II.3.7 in [1]) is sectionally complemented but it is not uniform. The lattices of Figure 1 and 3 are uniform but not sectionally complemented.

The proof of this theorem is based on a new lattice construction that will be described in Section 2. Further specializing the construction, in Section 3, we determine the congruence lattice of the new construct. In Section 4, we introduce a very simple kind of chopped lattice, and in Section 5 we prove that the ideal lattice of this chopped lattice is uniform. The proof of the theorem is presented in Section 6. Finally, in Section 7, we point out that variants of Theorem 1.1 for infinite lattices are implicit in G. Grätzer and E. T. Schmidt [4], and we present some open problems.

For the common lattice theoretic concepts and notation, we refer the reader to G. Grätzer [1]. $\mathbf{B}_n$ will denote the Boolean lattice of 2^n elements. For a bounded lattice A with bounds 0 and 1, let $A^- = A - \{0, 1\}$.

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2. A lattice construction

Let A and B be lattices. Let us assume that A is bounded, with bounds 0 and 1, $0 \neq 1$. We introduce a new lattice construction $N(A, B)$.

For $u \in A \times B$, we use the notation $u = \langle u_A, u_B \rangle$; the binary relation $\leq_\times$ will denote the partial ordering on $A \times B$, and $\wedge_\times$, $\vee_\times$ the meet and join in $A \times B$,

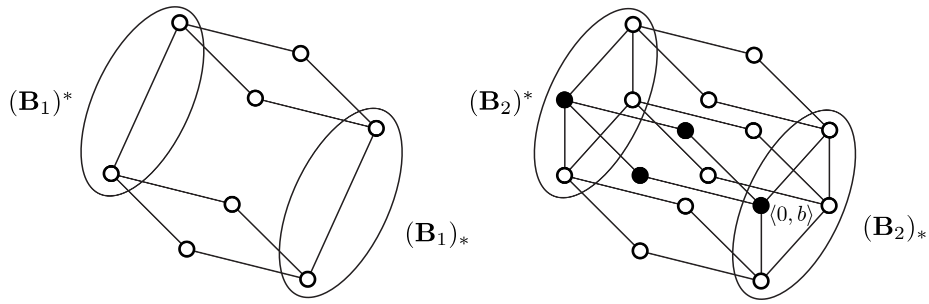

FIGURE 1. The lattices $N(\mathbf{B}_2, \mathbf{B}_1)$ and $N(\mathbf{B}_2, \mathbf{B}_2)$.

respectively. On the set $A \times B$, we define a binary relation $\leq_{N(A,B)}$ (denoted by $\leq_N$, if A and B are understood) as follows:

$$(1) \qquad \leq_N \;=\; \leq_\times \;-\{\,\langle u, v\rangle \mid u,\; v \in A^- \times B,\; u_B \neq v_B\,\}.$$

To help visualize this definition, Figure 1 shows the diagrams of $N(\mathbf{B}_2, \mathbf{B}_1)$ and $N(\mathbf{B}_2, \mathbf{B}_2)$.

Lemma 2.1. *$N(A, B)$ is partially ordered by $\leq_N$, in fact, $N(A, B)$ is a lattice. The meet and join in $N(A, B)$ of $\leq_N$-incomparable elements can be computed using the formulas:*

$$u \wedge_N v = \begin{cases} \langle 0, u_B \wedge v_B\rangle, & \text{if } u \wedge_\times v \in A^- \times B \text{ and } u_B \neq v_B;\\ u \wedge_\times v, & \text{otherwise.}\end{cases}$$

$$u \vee_N v = \begin{cases} \langle 1, u_B \vee v_B\rangle, & \text{if } u \vee_\times v \in A^- \times B \text{ and } u_B \neq v_B;\\ u \vee_\times v, & \text{otherwise.}\end{cases}$$

PROOF. Since $\leq_\times$ is reflexive, it follows that $\leq_N$ is reflexive, since $u_B \neq u_B$ fails, for all $u \in A \times B$.

Since $\leq_\times$ is antisymmetric, so is $\leq_N$.

Let u, v, $w \in A \times B$; let us assume that $u \leq_N v$ and $v \leq_N w$. Since $\leq_\times$ is transitive, we conclude that $u \leq_\times w$. So if $u \leq_N w$ fails, then u, $w \in A^- \times B$ and $u_B \neq w_B$. It follows that $v \in A^- \times B$ and either $u_B \neq v_B$ or $v_B \neq w_B$, contradicting that $u \leq_N v$ or $v \leq_N w$. So $\leq_\times$ is transitive.

Let u, $v \in A \times B$ be $\leq_N$-incomparable elements, and let t be a lower bound of u and v in $N(A, B)$.

Case 1. $u \wedge_\times v$ is not a lower bound of both u and v in $N(A, B)$.

If $u \wedge_\times v$ is not a lower bound of both u and v in $N(A, B)$, say, $u \wedge_\times v \nleq_N u$, then u, $u \wedge_\times v \in A^- \times B$ and $u_B \neq (u \wedge_\times v)_B$ (which is the same as $u_B \neq v_B$). Since $t \leq_\times u \wedge_\times v$, it follows that $t_B \leq (u \wedge_\times v)_B < u_B$, so $t \notin A^- \times B$ (otherwise, we would have $t \nleq_N u$). We conclude that $t = \langle 0, t_B \rangle$, so $t_B \leq u_B \wedge v_B$, which yields that $t \leq \langle 0, u_B \wedge v_B \rangle$.

So in Case 1, $u \wedge_N v = \langle 0, u_B \wedge v_B \rangle$.

Case 2. $u \wedge_\times v$ is a lower bound of both u and v in $N(A, B)$.

If $t \nleq_N u \wedge_\times v$, then t, $u \wedge_\times v \in A^- \times B$ and $t_B < (u \wedge_\times v)_B$, so $u \in A^- \times B$ or $v \in A^- \times B$, say, $u \in A^- \times B$. Therefore, the assumption of Case 2, namely, $u \wedge_\times v \leq_N u$, implies that $(u \wedge_\times v)_B = u_B$. So t, $u \in A^- \times B$ and $t_B \neq u_B$, contradicting that $t \leq_N u$. Thus $t \nleq_N u \wedge_\times v$ leads to a contradiction. We conclude that $t \leq_N u \wedge_\times v$.

So in Case 2, $u \wedge_N v = u \wedge_\times v$.

Since the two cases correspond to the two clauses of the meet formula, this verifies the meet formula.

The join formula follows by duality. □

We shall use the notation: $B_* = \{0\} \times B$, $B^* = \{1\} \times B$, and for $b \in B$, $A_b = A \times \{b\}$. Note that B_* is an ideal and B^* is a dual ideal of $N(A, B)$. In Figure 1, we show the diagrams of $N(\mathbf{B}_2, \mathbf{B}_1)$ and $N(\mathbf{B}_2, \mathbf{B}_2)$; in both diagrams B_* and B^* are marked. In the diagram of $N(\mathbf{B}_2, \mathbf{B}_2)$, the black filled elements form $A_b = (\mathbf{B}_2)_b$, where b is an atom of $B = \mathbf{B}_2$.

Craig Platt suggested that there is a more intuitive way of looking at this construction. Let U be a lattice with a prime interval $[p, q]$. We create a new lattice U' by inserting into U a copy of A, identifying 0 with p and 1 with q. More formally, we define a partial ordering $\leq_{U'}$ on this new set U' as follows:

(i) $u \leq_{U'} v$ iff $u \leq_U v$, for u, $v \in U$.
(ii) $u \leq_{U'} v$ iff $u \leq_A v$, for u, $v \in A$.
(iii) $u \leq_{U'} v$ iff $u \leq_U p$, for $u \in U$ and $v \in A$.
(iv) $u \leq_{U'} v$ iff $q \leq_U v$, for $u \in A$ and $v \in U$.

It is easy to see that U' is a lattice and to get formulas for meets and joins, similar to those in Lemma 2.1. We can then describe the $N(A, B)$ construction as follows. We form $\{0, 1\} \times B$, where we regard $\{0, 1\}$ as the two-element lattice. In $\{0, 1\} \times B$, for any $b \in B$, we form the prime interval $[\langle 0, b \rangle, \langle 1, b \rangle]$ and insert a copy of A, identifying 0 with $\langle 0, b \rangle$ and 1 with $\langle 1, b \rangle$. As the next step, we can now verify that the partial ordering of this new construct is the same as the partial ordering described in (1), and then verify that Lemma 2.1 holds.

3. Congruences on $N(A,B)$.

Let K and L be lattices, and let α be an embedding of K into L. Given a congruence Θ of L, we can define the congruence Θ_1 on K *via* α, that is, for a, $b \in K$,

$$a \equiv b \quad (\Theta_1) \qquad \text{iff} \qquad a\alpha \equiv b\alpha \quad (\Theta).$$

We shall call Θ_1 the *restriction of* Θ *transferred* via *the isomorphism* α *to* K.

Let Ψ be a congruence relation of $N = N(A,B)$. Using the natural isomorphisms of B into $N(A,B)$ with images B_* and B^*, we define Φ_* and Φ^* as the restriction of Ψ to B_* and B^*, respectively, transferred *via* the natural isomorphisms to B. Let Θ_b as the restriction of Ψ to A_b, for $b \in B$ transferred *via* the natural isomorphisms to A.

Lemma 3.1. $\Phi_* = \Phi^*$.

Proof. Indeed, if $b_0 \equiv b_1$ (Φ_*), then $\langle 0, b_0\rangle \equiv \langle 0, b_1\rangle$ (Ψ). Joining both sides with $\langle 1, b_0 \wedge b_1\rangle$, we obtain that $\langle 1, b_0\rangle \equiv \langle 1, b_1\rangle$ (Ψ), that is, $b_0 \equiv b_1$ (Φ^*). And symmetrically. ☐

It is easy to see that $\Phi = \Phi_* = \Phi^* \in \operatorname{Con} B$, and $\{\, \Theta_b \mid b \in B \,\} \subseteq \operatorname{Con} A$ describe Ψ, but it is difficult to obtain a description of the congruences of $N(A,B)$, in general.

Let A be a bounded lattice. A congruence Θ of A *separates* 0, if $[0]\Theta = \{0\}$, that is, $x \equiv 0$ (Θ) implies that $x = 0$. Similarly, a congruence Θ of A *separates* 1, if $[1]\Theta = \{1\}$, that is, $x \equiv 1$ (Θ) implies that $x = 1$. We call the lattice A *non-separating*, if 0 and 1 are not separated by any congruence $\Theta \neq \omega$.

Lemma 3.2. *Let A and B be lattices with $|A| > 2$ and $|B| > 1$; let A be bounded, with bounds* 0 *and* 1. *Let us further assume that A is non-separating. Then the map sending $\Psi \neq \omega_N$ to its restriction to B_* transferred to B by the natural isomorphism is a bijection between the non-ω_N congruences of $N(A,B)$ and the congruences of B. Therefore,* $\operatorname{Con} N(A,B)$ *is isomorphic to* $\operatorname{Con} B$ *with a new zero added.*

Proof. Let $\Psi \neq \omega_N$ be a congruence relation of $N(A,B)$. We start with the following statement:

Claim 1. *There are elements $a_1 < a_2$ in A and an element $b_1 \in B$ such that*

$$\langle a_1, b_1\rangle \equiv \langle a_2, b_1\rangle \quad (\Psi).$$

Proof. Assume that $\langle u_1, v_1\rangle \equiv \langle u_2, v_2\rangle$ (Ψ) with $\langle u_1, v_1\rangle <_N \langle u_2, v_2\rangle$. We distinguish two cases.

First case: $u_1 = u_2$.

Then $v_1 < v_2$ and either $u_1 = u_2 = 0$ or $u_1 = u_2 = 1$. So either $\langle 0, v_1\rangle \equiv \langle 0, v_2\rangle$ (Ψ) or $\langle 1, v_1\rangle \equiv \langle 1, v_2\rangle$ (Ψ). By Lemma 3.1, either one of these congruences implies the other, so both of these congruences hold. Since $|A| > 2$, we can pick an $a \in A^-$. Then

$$\langle a, v_1\rangle = \langle a, v_1\rangle \vee \langle 0, v_1\rangle \equiv \langle a, v_1\rangle \vee \langle 0, v_2\rangle = \langle 1, v_2\rangle \quad (\Psi),$$

from which we conclude that $\langle a, v_1\rangle \equiv \langle 1, v_1\rangle$ (Ψ), so the claim is verified with $a_1 = a$, $a_2 = 1$, and $b_1 = v_1$.

Second case: $u_1 < u_2$.

Since we have assumed that $\langle u_1, v_1\rangle <_N \langle u_2, v_2\rangle$, it follows from the definition of $\leq_N$ that either $v_1 = v_2$, or $u_1 = 0$, or $u_2 = 1$.

If $v_1 = v_2$, then $\langle u_1, v_1\rangle \equiv \langle u_2, v_1\rangle$ (Ψ), so the claim is verified with $a_1 = u_1$, $a_2 = u_2$, and $b_1 = v_1$.

If $u_1 = 0$, then $\langle 0, v_2\rangle \equiv \langle u_2, v_2\rangle$ (Ψ), so the claim is verified with $a_1 = 0$, $a_2 = u_2$, and $b_1 = v_2$.

If $u_2 = 1$, then $\langle u_1, v_1\rangle \equiv \langle 1, v_1\rangle$ (Ψ), so the claim is verified with $a_1 = u_1$, $a_2 = 1$, and $b_1 = v_1$. ☐

Claim 2. *There is an element $b_2 \in B$ such that A_{b_2} is in a single congruence class of Ψ.*

Proof. By Claim 1, there are $a_1 < a_2$ in A and $b_1 \in B$ such that

$$\langle a_1, b_1\rangle \equiv \langle a_2, b_1\rangle \quad (\Psi).$$

Since A is non-separating, there exists $a_3 \in A$ with $0 < a_3$ and $0 \equiv a_3$ $(\Theta(a_1, a_2))$. Moreover, A_{b_1} is a sublattice of $N(A, B)$, so it follows that $\langle 0, b_1\rangle \equiv \langle a_3, b_1\rangle$ $(\Theta(\langle a_1, b_1\rangle, \langle a_2, b_1\rangle))$, and so

$$\langle 0, b_1\rangle \equiv \langle a_3, b_1\rangle \quad (\Psi).$$

So for any $b_2 \in B$ with $b_1 < b_2$, joining both sides with $\langle 0, b_2\rangle$, we obtain that $\langle 0, b_2\rangle \equiv \langle 1, b_2\rangle$ (Ψ), that is, A_{b_2} is in a single Ψ class.

This completes the proof of the claim, unless b_1 is the unit element, 1_B, of B. In this case,

$$\langle 0, 1_B\rangle \equiv \langle a_3, 1_B\rangle \quad (\Psi).$$

Since A is non-separating, there exists $a_4 \in A$ with $a_4 < 1$ and $a_4 \equiv 1$ $(\Theta(0, a_3))$. Moreover, A_{1_B} is a sublattice of $N(A, B)$, so it follows that $\langle a_4, 1_B\rangle \equiv \langle 1, 1_B\rangle$ $(\Theta(\langle 0, 1_B\rangle, \langle a_3, 1_B\rangle))$, and so

$$\langle a_4, 1_B\rangle \equiv \langle 1, 1_B\rangle \quad (\Psi).$$

Now choose any $b_2 < 1_B$ (recall that we have assumed that $|B| > 1$). Meeting both sides with $\langle 1, b_2\rangle$, we obtain that

$$\langle 1, b_2\rangle \equiv \langle 0, b_2\rangle \quad (\Psi),$$

that is, A_{b_2} is in a single congruence class of Ψ. □

Claim 3. *A_b is in a single congruence class of Ψ, for each $b \in B$.*

Proof. Let $b \in B$. By Claim 2, there is an element $b_2 \in B$ such that A_{b_2} is in a single congruence class of Ψ, that is,

$$\langle 1, b_2\rangle \equiv \langle 0, b_2\rangle \quad (\Psi).$$

Therefore,

$$\langle 1, b\rangle = (\langle 1, b_2\rangle \vee \langle 0, b \vee b_2\rangle) \wedge \langle 1, b\rangle \equiv (\langle 0, b_2\rangle \vee \langle 0, b \vee b_2\rangle) \wedge \langle 1, b\rangle = \langle 0, b\rangle \quad (\Psi),$$

that is, A_b is in a single congruence class of Ψ. □

Now the statement of the lemma is easy to verify. It is clear that the map $\Psi \mapsto \Phi$ is one-to-one, for $\Psi \in \operatorname{Con} N(A,B) - \{\omega_N\}$. It is also onto: given a congruence Φ of B, define Ψ on $N(A,B)$ by

$$\langle u_1, v_1\rangle \equiv \langle u_2, v_2\rangle \quad (\Psi) \qquad \text{iff} \qquad v_1 \equiv v_2 \quad (\Phi).$$

Then Ψ is in $\operatorname{Con} N(A,B) - \{\omega_N\}$ and it maps to Φ. □

4. Chopped lattices

Let M be a finite poset such that $\inf\{a, b\}$ exists in M, for all a, $b \in M$. We define in M:

$$\begin{aligned} a \wedge b &= \inf\{a, b\}, && \text{for all } a,\ b \in M; \\ a \vee b &= \sup\{a, b\}, && \text{whenever } \sup\{a, b\} \text{ exists.} \end{aligned}$$

This makes M into a finite *chopped lattice*.

An equivalence relation Θ on the chopped lattice M is a *congruence relation* iff, for all a_0, a_1, b_0, $b_1 \in M$,

$$\begin{aligned} a_0 &\equiv b_0 \quad (\Theta), \\ a_1 &\equiv b_1 \quad (\Theta) \end{aligned}$$

imply that

$$\begin{aligned} a_0 \wedge a_1 &\equiv b_0 \wedge b_1 \quad (\Theta), \\ a_0 \vee a_1 &\equiv b_0 \vee b_1 \quad (\Theta), \quad \text{whenever } a_0 \vee a_1 \text{ and } b_0 \vee b_1 \text{ both exist.} \end{aligned}$$

The set $\operatorname{Con} M$ of all congruence relations of M is a lattice.

An *ideal* I of a finite chopped lattice M is a non-empty subset $I \subseteq M$ with the following two properties:

(i) $i \wedge a \in I$, for $i \in I$ and $a \in M$;
(ii) $i \vee j \in I$, for $i, j \in I$, provided that $i \vee j$ exists in M.

The ideals of the finite chopped lattice M form the finite lattice $\operatorname{Id} M$.

The following lemma was published in G. Grätzer [1].

Lemma 4.1 (G. Grätzer and H. Lakser)**.** *Let M be a finite chopped lattice. Then for every congruence relation Θ of M, there exists exactly one congruence relation $\overline{\Theta}$ of $\operatorname{Id} M$ such that, for $a, b \in M$,*

$$(a] \equiv (b] \ (\overline{\Theta}) \quad \textit{iff} \quad a \equiv b \ (\Theta).$$

In particular, $\operatorname{Con} M \cong \operatorname{Con}(\operatorname{Id} M)$.

We shall need a very simple type of finite chopped lattices. Let C and D be finite lattices such that $J = C \cap D$ is an ideal in C and in D; let m denote the generator of J. Then, with the natural partial ordering, $M(C, D) = C \cup D$ is a finite chopped lattice. Note that if $a \vee b = c$ in $M(C, D)$, then either $a, b, c \in C$ and $a \vee b = c$ in C or $a, b, c \in D$ and $a \vee b = c$ in D.

We can coordinatize $\operatorname{Id} M(C, D)$ as follows:

Lemma 4.2. *Let $\overline{M}(C, D) = \{\, \langle x, y \rangle \in C \times D \mid x \wedge m = y \wedge m \,\}$, a subposet of $C \times D$. Then $\overline{M}(C, D)$ is a finite lattice and $\operatorname{Id} M(C, D) \cong \overline{M}(C, D)$.*

Proof. This is obvious since an ideal I of $M(C, D)$ can be written uniquely in the form $I_C \cup I_D$, where I_C is an ideal of C and I_D is an ideal of D satisfying $I_C \cap J = I_D \cap J$. Let $I_C = (x]$ and $I_D = (y]$. Then $I_C \cap J = I_D \cap J$ translates to $x \wedge m = y \wedge m$. The map $I \mapsto \langle x, y \rangle$ is the required isomorphism. □

The following is an obvious application of the coordinatization.

Lemma 4.3. *Let U be an ideal of C and let V be an ideal of D. Let us regard $U \cup V$ as a subset of $\operatorname{Id} M(C, D)$ by identifying an element with the principal ideal it generates. If $U \cap V = \{0\}$, then the sublattice generated by $U \cup V$ in $\operatorname{Id} M(C, D)$ is an ideal, and it is isomorphic to $U \times V$.*

The congruences of $M(C, D)$ are easy to describe. Let Λ be a congruence of the chopped lattice $M(C, D)$, and let Λ_C and Λ_D be the restrictions of Λ to C and D, respectively. Then Λ_C is a congruence of C and Λ_D is a congruence of D satisfying the condition: Λ_C restricted to J equals Λ_D restricted to J. Conversely, if Θ is a congruence on C and Ψ is a congruence on D satisfying that Θ restricted to J equals Ψ restricted to J (in formula, $\Theta\rceil_J = \Psi\rceil_J$), then we can define a congruence Λ on $M(C, D)$ as follows:

(i) $x \equiv y \ (\Lambda)$ iff $x \equiv y \ (\Theta)$, for x, $y \in C$.
(ii) $x \equiv y \ (\Lambda)$ iff $x \equiv y \ (\Psi)$, for x, $y \in D$.
(iii) $x \equiv y \ (\Lambda)$ iff $x \equiv x \wedge y \ (\Theta)$ and $y \equiv x \wedge y \ (\Psi)$, for $x \in C$ and $y \in D$, and symmetrically.

We summarize:

Lemma 4.4. *Let C and D be finite lattices such that $J = C \cap D = (m]$ is an ideal in C and in D. Then*

$$\operatorname{Con}\operatorname{Id} M(C, D) \cong \{\, \langle\Theta, \Psi\rangle \in \operatorname{Con} C \times \operatorname{Con} D \mid \Theta\rceil_J = \Psi\rceil_J \,\}.$$

Now let U be a finite lattice with an ideal V isomorphic to $\mathbf{B}_n$. We identify V with the ideal $(\mathbf{B}_n)_* = (\langle 0, 1\rangle]$ of $N(\mathbf{B}_2, \mathbf{B}_n)$ to obtain the chopped lattice $K = M(U, N(\mathbf{B}_2, \mathbf{B}_n))$. Let m denote the generator of $V = (\mathbf{B}_n)_*$. Then $\operatorname{Id} K \cong \overline{M}(U, N(\mathbf{B}_2, \mathbf{B}_n))$.

Lemma 4.5. *Let $u \in U$. Then*

$$\{\, y \in N(\mathbf{B}_2, \mathbf{B}_n) \mid \langle u, y\rangle \in \overline{M}(U, N(\mathbf{B}_2, \mathbf{B}_n)) \,\}$$

is isomorphic to $\mathbf{B}_2$.

Proof. This is clear, since there are exactly four elements y of $N(\mathbf{B}_2, \mathbf{B}_n)$ satisfying that $u \wedge m = y \wedge m$, namely, the elements of $(\mathbf{B}_2)_{u \wedge m}$, and they form a sublattice isomorphic to $\mathbf{B}_2$. Therefore,

$$\{\, y \in N(\mathbf{B}_2, \mathbf{B}_n) \mid \langle u, y\rangle \in \overline{M}(U, N(\mathbf{B}_2, \mathbf{B}_n)) \,\}$$

is a four-element set, closed under coordinate-wise meets and joins; the statement follows. ☐

5. Congruence classes

Let U and V be given as at the end of the previous section. Let us further assume that U is uniform. Let K be the chopped lattice $M(U, N(\mathbf{B}_2, \mathbf{B}_n))$ and we identify $\operatorname{Id} K$ with $\overline{M}(U, N(\mathbf{B}_2, \mathbf{B}_n))$.

Lemma 5.1. $\operatorname{Id} K$ *is uniform.*

Proof. A congruence Λ of $\operatorname{Id} K$ can be described by Lemma 4.4 by a pair of congruences $\langle\Theta, \Psi\rangle$, where Θ is a congruence of U, Ψ is a congruence of $N(\mathbf{B}_2, \mathbf{B}_n)$, and Θ and Ψ restrict to the same congruence of $V = (\mathbf{B}_n)_*$.

The trivial congruences, $\omega_{\operatorname{Id} K} = \langle\omega_U, \omega_{N(\mathbf{B}_2, \mathbf{B}_n)}\rangle$ and $\iota_{\operatorname{Id} K} = \langle\iota_U, \iota_{N(\mathbf{B}_2, \mathbf{B}_n)}\rangle$, are obviously uniform. Therefore, we need only look at two cases.

First case: Λ is represented by $\langle \Theta, \omega \rangle$.

So $\Theta\rceil_V = \omega_V$. Let $\langle x, y \rangle$ be an element of $\operatorname{Id} K$ and note that

$$[\langle x, y \rangle]\langle \Theta, \omega \rangle = \{\, \langle t, y \rangle \in \operatorname{Id} K \mid t \equiv x \,(\Theta) \,\}.$$

If $t \equiv x \;(\Theta)$, then $t \wedge m \equiv x \wedge m \;(\Theta)$, but $\Theta\rceil_V = \omega_V$, so $t \wedge m = x \wedge m$. We conclude that

$$[\langle x, y \rangle]\langle \Theta, \omega \rangle = \{\, \langle t, y \rangle \mid t \equiv x \,(\Theta) \,\},$$

and

$$|[\langle x, y \rangle]\langle \Theta, \omega \rangle| = |[x]\Theta|.$$

So Λ is uniform; each congruence class of Λ is of the same size as a congruence class of Θ.

Second case: Λ is represented by $\langle \Theta, \Psi \rangle$, where $\Psi \neq \omega$.

Let $\langle x, y \rangle$ be an element of $\operatorname{Id} K$. Then

$$[\langle x, y \rangle]\langle \Theta, \Psi \rangle = \{\, \langle w, z \rangle \in \operatorname{Id} K \mid x \equiv w \,(\Theta) \text{ and } y \equiv z \,(\Psi) \,\}.$$

For a given w, if $\langle w, t_1 \rangle$ and $\langle w, t_2 \rangle \in \operatorname{Id} K$, then $t_1 \equiv t_2 \;(\Psi)$ because $(\mathbf{B}_2)_w$ is in a single congruence class of Ψ by Lemma 3.2 (in particular, by Claim 3) and

$$\{\, t \in N(\mathbf{B}_2, \mathbf{B}_n) \mid \langle w, t \rangle \in \operatorname{Id} K \,\} = (\mathbf{B}_2)_{w \wedge m}$$

by Lemma 4.5, so

$$|\{\, t \in N(\mathbf{B}_2, \mathbf{B}_n) \mid \langle w, t \rangle \in \operatorname{Id} K \,\}| = |(\mathbf{B}_2)_{w \wedge m}| = 4.$$

We conclude that

$$[\langle x, y \rangle]\langle \Theta, \Psi \rangle = \{\, \langle w, z \rangle \in \operatorname{Id} K \mid x \equiv w \,(\Theta),\ z \in (\mathbf{B}_2)_{w \wedge m} \,\},$$

and

$$|[\langle x, y \rangle]\langle \Theta, \Psi \rangle| = 4|[x]\Theta|.$$

So Λ is uniform; each congruence class of Λ is four-times the size of a congruence class of Θ. $\square$

6. Proving the theorem

We shall prove Theorem 1.1 by induction. To make the induction work, we need to restrict ourselves to a much narrower class of lattices. We define this class with the following two properties of a finite lattice L:

(P) Every join-irreducible congruence of L is of the form $\Theta(0,p)$, for a suitable atom p of L.
(Q) If $\Theta_1, \Theta_2, \dots, \Theta_n \in \mathrm{J}(\mathrm{Con}\, L)$ are pairwise incomparable, then L contains atoms $p_1, p_2, \dots, p_n$ that generate an ideal isomorphic to $\mathbf{B}_n$ and satisfy $\Theta_i = \Theta(0, p_i)$, for all $i \leq n$.

Now we are ready to formulate the stronger form of Theorem 1.1, and prove it.

Theorem 6.1. *For any finite distributive lattice D, there exists a finite uniform lattice L such that the congruence lattice of L is isomorphic to D, and L satisfies properties* (P) *and* (Q)*.*

Proof. Let D be a finite distributive lattice with n join-irreducible elements.

If $n = 1$, then $D \cong \mathbf{B}_1$, so there is a lattice $L = \mathbf{B}_1$ that satisfies Theorem 6.1.

Let us assume that, for all finite distributive lattices with fewer than n join-irreducible elements, there exists a lattice L satisfying Theorem 6.1 and properties (P) and (Q).

Let q be a minimal element of $\mathrm{J}(D)$ and let $q_1, \dots, q_k$ $(k \geq 0)$ list all upper covers of q in $\mathrm{J}(D)$. Let D_1 be a distributive lattice with $\mathrm{J}(D_1) = \mathrm{J}(D) - \{q\}$. By the inductive assumption, there exists a lattice L_1 satisfying $\mathrm{Con}\, L_1 \cong D_1$ and (P) and (Q).

If $k = 0$, then $D \cong \mathbf{B}_1 \times D_1$, and so $L = \mathbf{B}_1 \times L_1$ obviously satisfies all the requirements of the theorem. So we assume that $1 \leq k$.

The congruences of L_1 corresponding to the q_i's are pairwise incomparable and therefore can be written in the form $\Theta(0, p_i)$ and the p_i's generate an ideal I_1 isomorphic to $\mathbf{B}_k$. The lattice $N(\mathbf{B}_2, \mathbf{B}_k)$ also contains an ideal $(\mathbf{B}_k)_*$ isomorphic to $\mathbf{B}_k$. Identifying I_1 and $(\mathbf{B}_k)_*$, we get the chopped lattice K and the lattice $L = \mathrm{Id}\, K$. The chopped lattice K is sketched in Figure 2. L is uniform by Lemma 5.1.

Let Θ be a join-irreducible congruence of L. Then we can write Θ as $\Theta(a,b)$, where a is covered by b. From Lemma 4.2 it follows that we can assume that either a, $b \in L_1$ or a, $b \in N(\mathbf{B}_2, \mathbf{B}_k)$. In either case, we find an atom q in L_1 or in $N(\mathbf{B}_2, \mathbf{B}_k)$, so that $\Theta(a,b) = \Theta(0,q)$ in L_1 or in $N(\mathbf{B}_2, \mathbf{B}_k)$. Obviously, q is an atom of L and $\Theta(a,b) = \Theta(0,q)$ in L, verifying (P) for L.

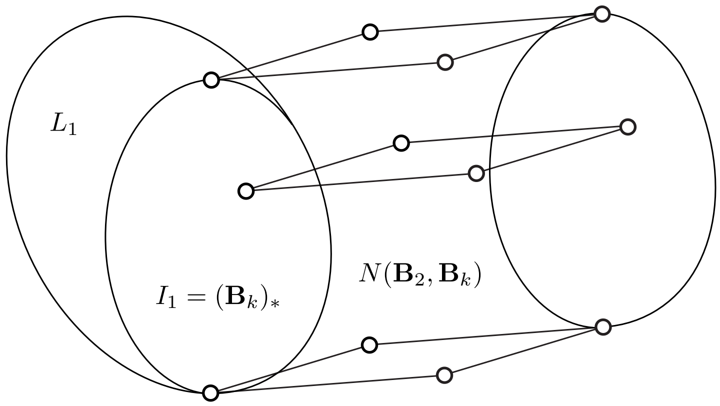

FIGURE 2. The chopped lattice K.

Let Θ_1, Θ_2, ..., Θ_t be pairwise incomparable join-irreducible congruences of L. To verify condition (Q), we have to find atoms p_1, p_2, ..., p_t of L satisfying $\Theta_i = \Theta(0, p_i)$, for all $i \leq t$, and such that p_1, p_2, ..., p_t generate an ideal of L isomorphic to $\mathbf{B}_t$.

Let p denote an atom in $N(\mathbf{B}_2, \mathbf{B}_k) - I_1$; there are two, but they generate the same congruence $\Theta(0, p)$. If $\Theta(0, p)$ is not one of Θ_1, Θ_2, ..., Θ_t, then clearly we can find p_1, p_2, ..., p_t in L_1 as required, and p_1, p_2, ..., p_t also serves in L.

If $\Theta(0, p)$ is one of Θ_1, Θ_2, ..., Θ_t, say, $\Theta(0, p) = \Theta_t$, then let p_1, p_2, ..., p_{t-1} be the set of atoms establishing (Q) for Θ_1, Θ_2, ..., Θ_{t-1} in L_1 and therefore, in L. Then $p_1, p_2, \ldots, p_{t-1}, p$ represent the congruences Θ_1, Θ_2, ..., Θ_t and they generate an ideal isomorphic to $\mathbf{B}_t$ by Lemma 4.3. Therefore, L satisfies (Q).

Finally, it is clear from this discussion that $\mathrm{J}(\mathrm{Con}\, K)$ has exactly one more element than $\mathrm{J}(\mathrm{Con}\, L_1)$, namely, $\Theta(0, p)$, and this join-irreducible congruence relates to the join-irreducible congruences of $\mathrm{Con}\, L_1$ exactly as q relates to the join-irreducible elements of D. Therefore, $D \cong \mathrm{Con}\, L$. ☐

7. Concluding comments

7.1. **Infinite lattices.** It is natural to ask whether Theorem 1.1 can be extended to infinite lattices. Since we do not know which distributive lattices can be represented as congruence lattices of lattices, the following concept is helpful in solving this problem.

Let L be a lattice. A lattice K is a *congruence-preserving extension* of L, if K is an extension and every congruence of L has *exactly one* extension to K.

Of course, then the congruence lattice of L is isomorphic to the congruence lattice of K.

We can prove the following two results:

Theorem 7.1. *Every infinite lattice L has a uniform congruence-preserving extension K.*

Theorem 7.2. *Every infinite lattice L with zero has a uniform congruence-preserving $\{0\}$-preserving extension K.*

Let us call a lattice K *strongly uniform*, if $|A| = |K|$ holds for every congruence $\Theta > \omega$ of K and every congruence class A of Θ.

Let us call a lattice K *regular*, if whenever Θ and Φ are congruences of K and Θ and Φ share a congruence class, then $\Theta = \Phi$.

Theorems 7.1 and 7.2 were proved, but not stated, in G. Grätzer and E. T. Schmidt [4] in the following stronger form:

Theorem 7.3. *Every infinite lattice L has a strongly uniform and regular congruence-preserving extension K.*

Theorem 7.4. *Every infinite lattice L with zero has a strongly uniform and regular congruence-preserving $\{0\}$-preserving extension K.*

We prove Theorem 7.3 in [4] (without stating that K is strongly uniform), by forming a transfinite sequence $\langle \langle a_\gamma, b_\gamma, c_\gamma \rangle \mid \gamma < \alpha \rangle$ with the following properties:

(i) α is a limit ordinal.
(ii) $\langle a_\gamma, b_\gamma, c_\gamma \rangle \in L^3$ and $a_\gamma < b_\gamma$, for $\gamma < \alpha$.
(iii) Every $\langle a, b, c \rangle \in L^3$ with $a < b$ occurs as $\langle a_\gamma, b_\gamma, c_\gamma \rangle$, for some $\gamma < \alpha$.

Then we construct a direct union of lattices as follows:

Let L_0 be $\mathbf{M}_3\langle L \rangle$, with L identified with $L\varphi_{a_0}$, where $\mathbf{M}_3\langle L \rangle$ is the boolean triple construction of G. Grätzer and F. Wehrung [5] and φ_{a_0} is the map $x \mapsto \langle x, a_0, x \wedge a \rangle$.

If $\gamma = \delta + 1$ and L_δ is defined, then let $L_\gamma = \mathbf{M}_3\langle L_\delta \rangle$, where L is identified in $\mathbf{M}_3\langle L_\delta \rangle$ with $L_\delta \varphi_{a_\gamma}$, where φ_{a_γ} is the map $x \mapsto \langle x, a_\gamma, x \wedge a \rangle$.

If γ is a limit ordinal and L_δ is defined, for all $\delta < \gamma$, then we form the congruence-preserving extension $\bigcup(L_\delta \mid \delta < \gamma)$ of L, where L is identified with a sublattice of $\bigcup(L_\delta \mid \delta < \gamma)$ in the obvious fashion. Then define

$$L_\gamma = \mathbf{M}_3\langle \bigcup(L_\delta \mid \delta < \gamma) \rangle,$$

and let L be identified in L_γ with $L\varphi_{a_\gamma}$, where φ_{a_γ} is the map $x \mapsto \langle x, a_\gamma, x \wedge a_\gamma \rangle$.

Repeating this construction $|L|$ times (this is a slight change from [4]), we obtain the lattice K with $\alpha = |L|$.

In [4], we verify that K is regular. To see that K is strongly uniform, let $\Theta > \omega$ be a congruence of L and let A be a congruence class of Θ. Then there is a unique extension Θ_γ of Θ to a congruence of L_γ, and a unique extension A_γ of A to a congruence class of Θ_γ. Since $\Theta > \omega$, there are $a < b$ in L with $a \equiv b \pmod{\Theta}$. For every $c \in A$, by Lemma 3 of [4], $e = \langle c, b, c \wedge b \rangle \in A_{\gamma+1} - A_\gamma$, so we conclude that $|A_\alpha| \geq |L|$. Since $|K| = |L|$, we conclude that K is strongly uniform.

If L has a zero, to prove Theorem 7.4, we proceed as in Section 4 of [4], using the construction $\mathbf{M}_3\langle L, a \rangle$ rather than the construction $\mathbf{M}_3\langle L \rangle$.

7.2. **Problems.** In contrast with the results in Section 7.1, for finite lattices the existence of uniform congruence-preserving extensions remains open:

Problem 1. *Does every finite lattice K have a congruence-preserving extension to a finite uniform lattice L?*

Let L be a finite uniform lattice and let φ be an isomorphism between a finite distributive lattice D and $\operatorname{Con} L$. Then we can introduce a function $s = s(L, \varphi)$ from D to the natural numbers, as follows: Let $d \in D$; then $d\varphi$ is a congruence of L. Since L is uniform, all congruence classes of $d\varphi$ are of the same size. Let $s(d)$ be the size of the congruence classes.

The function s has the following obvious properties:

(s_1) $s(0) = 1$.
(s_2) If $a < b$ in D, then $s(a) < s(b)$.

Problem 2. *Characterize the function s. In other words, let f be a function from a finite distributive lattice D to the natural numbers that satisfies* (s_1) *and* (s_2) *above. What additional conditions on the function f are required for $f = s(L, \varphi)$, for some finite uniform lattice L and isomorphism $\varphi \colon D \to \operatorname{Con} L$?*

This problem may be too difficult to solve in its full generality. The following lists some special cases that may be easier to attack.

Problem 3. *Characterize the function s for some special classes of finite distributive lattices:*

(i) *Finite Boolean lattices.*
(ii) *Finite chains.*
(iii) *"Small" distributive lattices.*

A large number of papers were written on the size of the smallest lattice L that can represent with its congruence lattice a finite distributive lattice D. See Appendices A and C of G. Grätzer [1] for a detailed accounting. The best result is in G. Grätzer, H. Lakser, and E. T. Schmidt [2]:

Theorem. *Let D be a finite distributive lattice with n join-irreducible elements. Then there exists a finite (planar) lattice L with $O(n^2)$ elements such that the congruence lattice of L is isomorphic to D.*

As a final problem, we ask whether these investigations can be paralleled for uniform lattices.

Problem 4. *Let D be a finite distributive lattice with n join-irreducible elements. Is there a "small" uniform lattice L such that the congruence lattice of L is isomorphic to D?*

In G. Grätzer and E. T. Schmidt [4], a lattice L is called *homogeneous-regular*, if L is regular and for every congruence relation Θ of L, any two Θ classes are isomorphic, and the following problem is raised (Problem 10 of [4]):

Can every finite distributive lattice be represented as the congruence lattice of a homogeneous-regular (resp., very regular) finite lattice?

The result from [3], quoted in the Introduction, naturally raises the following question:

Problem 5. *Can every finite distributive lattice be represented as the congruence lattice of a finite, uniform, sectionally complemented lattice?*

We conclude this paper with Figure 3, which shows our construction of the uniform lattice L for the four-element chain D. The figure shows a congruence of L with two congruence classes of 16 elements each. These two congruence classes are obviously not isomorphic as lattices. So the present paper makes no contribution to the solution of Problem 10 of [4]. In connection with Problem 5, observe that the lattice L of Figure 3 is not sectionally complemented; it is not even even atomistic.

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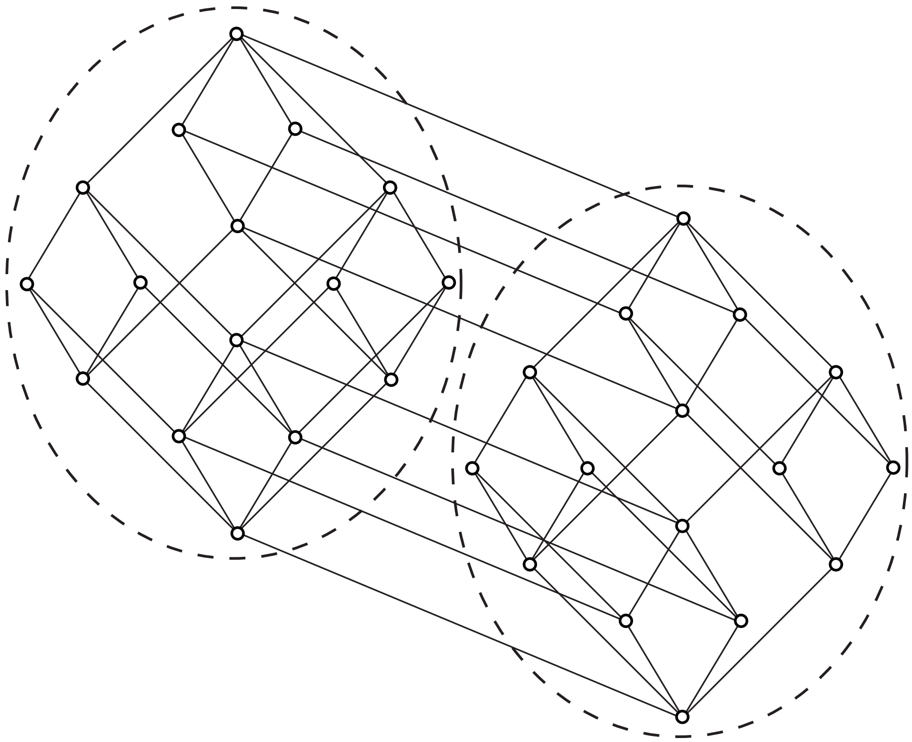

FIGURE 3. The lattice L representing the four-element chain.

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