A structure theorem of semimodular lattices and the Rubik's cube

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To the memory of my friends
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Abstract. In [4] we proved the following structure theorem: *every planar semimodular lattice is a special gluing, a patchwork of special intervals, called patch lattices*, shown in Figure 1. In this paper we characterize these patch lattices by invertible $(0, 1)$-matrices (see Theorem 2) and show that this structure theorem has a counterpart in the matrix theory (see Theorem 5). This gives hope that the theory of planar semimodular lattices can be traced back to matrix theory. We study the method of the transition to matrices in the planar case, which hopefully opens the way to handle higher dimensions easily to get a similar structure theorem.

1. Introduction

My conjecture is that the semimodular lattices have the following structure theorem: *every semimodular lattice is the patchwork of patch lattices.* I hope that this paper gives a guidance to solve the Conjecture 2 (probably via matrices), see at the and of this paper. This would be the first structure theorem of this kind in the theory of finite semimodular lattices. In the class of distributive lattices a good example is the Rubik's cube (this gave the idea to define the patchwork construction), where 27 small "unit" cubes are glued together by faces (we obtain every finite distributive lattices on this way from "unit" cubes). Our goal is to extend this construction to all semimodular lattice.

Mainly we study the two-dimensional case, but in many places we discuss the higher dimensional cases too order to clarify some basic concepts. You finde more reults in [17] and [18].

1.1. Source lattices. The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n\colon$ P has an n-element antichain$\}$. As usual, $\mathbf{J}(L)$ stands for the order of all nonzero join-irreducible elements of L. $\mathbf{Dim}(L) = \mathrm{w}(\mathbf{J}(L))$, consequently 2-dimensional means that the width of the order of join-irreducible elements is two. $\mathcal{C}_n$ denotes the chain $0 < 1 < .... < n-1$ of natural numbers. We define the source in subsection 2.3.

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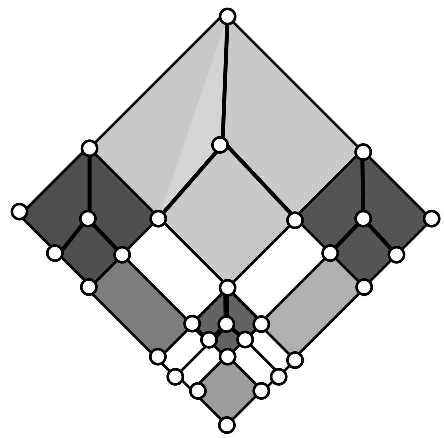


FIGURE 1. A patchwork in the two-dimensional case

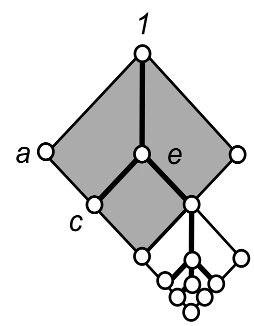


FIGURE 2. Patchwork of two patch lattices

Let us take the lattice N_7, the seven-element semimodular but not modular lattice. This is a special source lattice (an "elementary particle"), which is the smallest non distributive building stone of the 2D semimodular lattices.

Source lattices in higher dimensional cases, [17] are special join-homomorphic images of the direct powers of $\mathcal{C}_3$ and $\mathcal{C}_2$, which are the following semimodular lattices:

$$\mathbb{L}_{n,k} = ({\mathcal{C}_3}^{n-k} \times {\mathcal{C}_2}^{k})/\Phi,$$

where Φ is the cover-preserving join-congruence which has only one non-trivial congruence class T (called *beret*), this contains the dual atoms and the unit element. Every non-modular semimodular lattice contains as sublattice a source lattice $\mathbb{L}_{n,k}$. Then $\mathbb{L}_{2,0} \simeq N_7$ and $\mathbb{L}_{3,3} \simeq M_3$.

Two dimensional source lattices are: $\mathbb{L}_{2,0} \cong N_7$, $\mathbb{L}_{2,1} \cong {\mathcal{C}_2}^2$ and $\mathbb{L}_{2,2} \cong \mathcal{C}_2$. In Figure 3 we see $\mathbb{L}_{3,0}$. It is easy t see that $\mathbb{L}_{n,k}$ is a filter of $\mathbb{L}_{n,0}$.

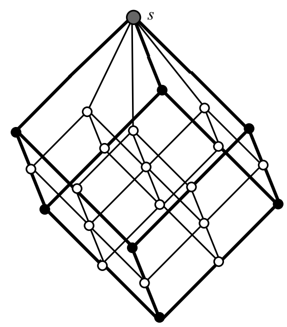


FIGURE 3. 3D source lattice, $\mathbb{L}_{3,0}$.

1.2. Patch lattices. Patch lattices are the building stones ("atoms") of the 2D semimodular lattices, see [4].

How can we derive the patch lattices from boolean lattices? This is the *nesting*, which is the following procedure: let L be a semimodular lattice and let I be an interval of L isomorphic to the 2^2-boolean lattice. We call this a 2-cell or *covering square*, see in section 2. On the other hand let us take the lattice N_7 and the four-element sublattice $\{a, b, 0, 1\}$ (see in Figure 3 and Figure 4,(II) the black marked circles) which is isomorphic to the 2^2-boolean lattice and it is called the *skeleton*, $\mathbf{Sk}(N_7)$ of N_7. There is an isomorphism to an isomorphism of $\mathbf{Sk}(N_7)$ onto an interval I' of a semimodular lattice $L_1 = L \cup I$ such that I and I' have the same bounds. There is a second construction which produces (IV) in Figure 3 from (III). We can repeat this construction for L_1 and a 2-cell then we get L_2, and so on, we get L_n.

On this way we get from the $L_0 \cong 2^2$ boolean lattice first N_7, these are the *patch lattices.* L_0 is a sublattice of L_n this is the *skeleton* of L_n Let us remark that the dual atoms of the skeleton are dual atoms of the patch lattices, see in Figure 6.

Lemma 1. *Every patch lattice has a skeleton.*

In [4] paper we used for nesting "adding fork to L". Fork is the order $\{c, d, e, 1\}$. In Figure 4 (V) and Figure 5 we see the nesting.

The two-dimensional semimodular lattices can be characterized by $(0, 1)$-matrices, $\mathbf{M}_L$, which determines L. The patch lattices are the semimodular lattices which are determined by special non singular (invertible) $(0, 1)$-matrices.

1.2.1. *The building tool: patchwork.* Let L and K be 2-dimensional lattices with the skeletons $\{a \wedge b, a, b, a \vee b\}$ resp. $\{c \wedge d, c, d, c \vee d\}$. The Hall-Dilworth gluing of L and K is called *patching* if $L \cap K \subset [a \wedge b, b]$ and $[c, c \vee d]$, (gluing over edges, $[a \wedge b, b]$ and $[c, c \vee d]$ are one-dimensional).

The following **structure theorem** was proved by G. Czédli and E. T. Schmidt [4].

Theorem 1. *Every two-dimensional semimodular lattice is the patchwork of patch lattices.*

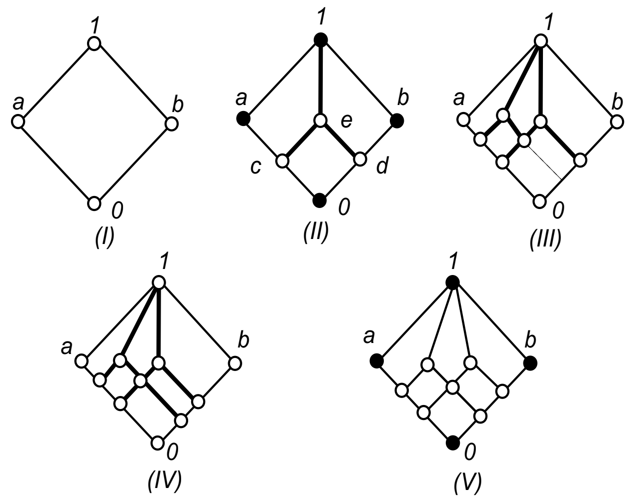


FIGURE 4. An example of nesting in the 2D case

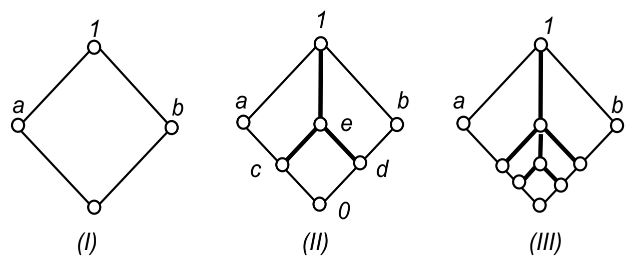


FIGURE 5. An other example of nesting in the 2D case

The structure of 2D semimodular lattices:

source lattices (”elementary particle”)
$\Downarrow$ *nesting* (spec. embedding)
patch lattices (”atoms”)
$\Downarrow$ *patching* (spec. gluing)
semimodular lattices

Remark. Similar theorem holds for *planar* semimodular lattices in this case the lattices M_n are patch lattices of dimension n.

1.3. Rectangular lattices. Rectangular lattices were introduced by Grätzer-Knapp [13] for planar semimodular lattices. This notion is an important tool by description (by the structure theorem, Grätzer-Knapp) of planar semimodular lattices. We define the rectangular lattices for arbitrary dimension.

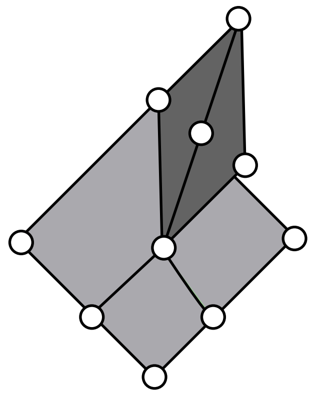


FIGURE 6. A Hall-Dilworth gluing which is not a patchwork

Definition 1. *A* rectangular *lattice L is a finite semimodular lattice in which $\mathbf{J}(L)$ is the disjoint sum of chains C_i.*

Geometric lattices are rectangular. In [10] we introduced the *almost geometric lattices* these are lattices in which $\mathbf{J}(L)$ is the disjoint (cardinal) sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the the direct products of chains. The lattices $M_3[\mathcal{C}_n]$ are modular, non distributive, rectangular 3D lattices, see in [16]). In Figure 14 is presented $M_3[\mathcal{C}_5]$.

To every 2D semimodular lattice L we assign a $(0,1)$-matrix $\mathbf{M}_L$ in which every row/column contains at most one "1" entry, see subsection 2.4. Let M be a square matrix such that in the last row and last column the entries are zeros. Deleting the last row and last column we get the the *restricted matrix* of M^-. Conversely, if N is a square matrix and we add a new last row and last column with zero entries this is called the *augmented matrix* of N^+. We will see that if $N = [1]$ then N^+ is the (augmented) matrix of N_7.

Since the matrices seem to the best tool to handle the semimodular lattices and in [4] we don't speak on matrices we define here additionally the patch matrices and extend Theorem 2 with condition (3).

Definition 2. *A* patch matrix *is a square* $(0,1)$*-matrix in which every row/column except the last row/column contains exactly one non-zero entry and in the last row/column all entries are 0.*

The following theorem is from [3], condition (3) is new.

Theorem 2. *Let L be a 2D rectangular semimodular lattice. The following three conditions are equivalent:*

(1) L is a patch lattice, i.e. a nested four-element boolean lattice,
(2) L has two dual atoms p and q such that $p \wedge q = 0$ (then $0, p, q, 1$ is the skeleton of L),
(3) $\mathbf{M}_L$ is a patch matrix.

The proof is in Section 3. The restricted matrix of $\mathbf{M}_L$ is a special non-singular (invertible) matrix. More equivalent conditions see in [4]

In the two and three dimensional cases $\mathbf{dim}(\mathbf{Sk}(\mathbb{L}_{m,0})) = \mathbf{dim}(\mathbb{L}_{m,0})$.

2. Transition from lattice to matrix.

2.1. Cover-preserving join-homomorphism. There is a trivial "representation theorem for finite lattices: each of them is a join-homomorphic image of a finite distributive lattice D. This follows from the fact that the finite free join semilattices with zero are the finite Boolean lattices.

The semimodular lattices are are very special join-homomorphic images of finite distributive lattices.

A planar lattice is called *slim* if every covering square is an interval. Now let L and K be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the cvering relation $\preceq$. Similarly, a joincongruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving.

Theorem 3. *(Manfred Stern's theorem,* [19]*) Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product $D = C_1 \times C_2 \times \ldots \times C_k$ of finite chains.*

In recent years it was found that this theorem has many interesting consequences. The direct product of n-chains can be considered as an **n-dimensional rectangular shape**, especially a boolean lattice with 2^n-elements is a n -dimensional cube, the direct product of two chains is geometrically a plain. This leads to a **geometrical approach** of the semimodular lattices, [17].

In a semimodular lattice the maximal chains have the same length. Assume that L is semimodular lattice and $a, b, u, v \in D, L = \varphi(D)$ and $u \leq a \prec b \leq v$. In this case φ has a special property. If E is a maximal chain of between u and v and $a, b \in E$, $\varphi(a) = \varphi(b)$ and F is an other maximal chain between u and v , then there exist $c, d \in F, c \prec d$ such that $\varphi(c) = \varphi(d)$. This property is just the cover-preserving property (this is not the usual form).

In [1] we proved:

Lemma 2. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

Stern's theorem was rediscovered by G. Czédli and E. T. Schmidt [1], see the following theorem (Stern's result was well-hidden in his book):

Theorem 4. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of finite chains, $C_1, C_2, \ldots, C_n$, these are maximal subchains of L, $n = \mathbf{Dim}(L) = \mathrm{w}(\mathbf{J}(L))$ such that $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup \ldots \cup C_n$.*

Let us recall the main result from Grätzer and Knapp [13]:

Corollary 1. *(Grätzer and Knapp [13]) Each finite planar semimodular lattice can be obtained from a cover-preserving join-homomorphic image of the direct product of two finite chains and adding doubly irreducible elements to the interiors of covering squares.*

2.2. The grid.

Definition 3. *A* grid *of a semimodular lattice L is $G = C_1 \times C_2 \times ... \times C_n$, where the C_i-s are maximal subchains of L, $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup ... \cup C_n$, $n = \mathbf{dim}(L)$.*

Observe that a grid can be considered as a coordinate system (in physics is called reference frame). If we consider a semimodular lattice L then we choose a grid G (i.e. the maximal chains $C_1, C_2, ...C_n$) and fix them.

By [1] L is the cover-preserving join-homomorphic image of G.

Remark 1. Let $D_1, D_2, ...D_n$, $n = \mathbf{dim}(L)$ be subchains of L such that $\mathbf{J}(L) = D_1 \cup D_2 \cup ... \cup D_n$ then $\underline{G} = D_1 \times ... \times D_n$ is called a (lower) *grid* of L.

2.3. The source. To describe the cover-preserving join- congruences of a distributive lattice G we need the notion of source elements of G. Czédli and E. T. Schmidt [3]. Let Θ be a cover-preserving join-congruence of G.

Definition 4. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t \ (\Theta)$ and for every prime quotient u/v $s/t \searrow u/v$, and $s \neq u$ imply $u \not\equiv v \ (\Theta)$. The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

Lemma 3. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s \ (\Theta)$. If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z \ (\Theta)$.*

Proof. Let s be a source element of Θ then $s \equiv t \ (\Theta)$ for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t \ (\Theta)$. This implies $x \wedge t \not\equiv t \ (\Theta)$, so have $x \equiv s \ (\Theta)$.

To prove that $v \not\equiv z \ (\Theta)$, we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely x, v, t.

These generate an eight-element covering boolean lattice in which $s \equiv t \ (\Theta)$, $s \equiv x \ (\Theta)$ and $s \equiv v \ (\Theta)$.

By the choice of t we know that $v \not\equiv v \wedge t \ (\Theta)$, $x \not\equiv x \wedge t \ (\Theta)$ and $z \not\equiv x \wedge t \wedge v$ (Θ). It follows that $x \not\equiv t \ (\Theta)$, otherwise by the transitivity $x \not\equiv v \ (\Theta)$. This implies $t \wedge x \not\equiv t \wedge x \wedge v \ (\Theta)$. Take the covering square $\{x \wedge v \wedge t, z, t \wedge x, x\}$ then by Lemma 2 $z \not\equiv x \ (\Theta)$, which implies $z \not\equiv v \ (\Theta)$. □

The following results are proved in [17]. The source $\mathcal{S}$ satisfies an independence property:

Definition 5. *Two elements s_1 and s_2 of a 2D-distributive lattice are s-*independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

Remark. In the 2D case every s-independent subset is the source of some cover-preserving join-congruence. In higher dimensional cases this is not true, we need an other property too, the shower property [17].

Lemma 4. *Every row/column contains at most one source element.*

The semimodular lattice L is determined by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset and therefore we write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Let Θ be a cover-preserving join-congruence of an 2-dimensional grid G and let $\mathcal{S}$ be the source of Θ. Take $\mathcal{S}$ and the set of all lower covers of the source elements ${s'}_i \prec s$ $(i \in \{1, 2, 3\})$. Then we have the following set of prime intervals of G:

$$P = \{[{s'}_i, s]|, s \in \mathcal{S}\}.$$

Let $\Theta_{\mathcal{S}}$ be the join congruence generated by this set of prime intervals, i.e. for a prime interval $[a, b]$ $a \equiv b$ (Θ_S) if and only if there is a $s \in \mathcal{S}$ prime interval $[{s'}_i, s]$ such that $[a, b]$ is upper perspective to a $[{s'}_i, s]$. Then $\Theta = \Theta_{\mathcal{S}}$ (if $\mathcal{S}$ is an s-independent set then Θ_S.

It is easy to prove that in the 2D case every s-independent subset $\mathcal{S}$ determinate a cover-preserving join-congruence Θ.

Lemma 5. *Let G be a 2-dimensional grid, i.e. the direct product of two chains. Let $\mathcal{S}$ be an s-independent subset of G. Then there exists a cover-preserving join-congruence Θ of G with the source $\mathcal{S}$.*

The meet of two cover-preserving join-congruence is in generally not cover-preserving.

Θ_s denotes the cover-preserving join-congruence determined by s, see in Figure 7. The source of Θ_s is $\{s\}$.

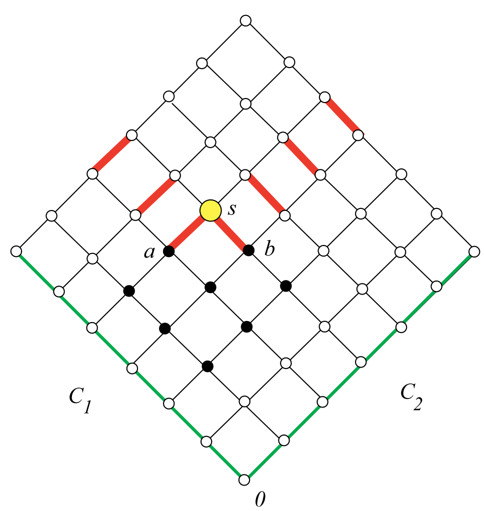


FIGURE 7. The join-congruence Θ_s

2.4. The matrix. To every semimodular lattices we can assigne $(0,1)$-matrices played the role of barcodes.

Let L be a 2D semimodular lattice. By Theorem 2 we have a grid $G = \mathcal{C}_k{}^n$ (which will be fixed) and a cover-preserving join-congruence Θ of G such that $G/\Theta \cong L$. In Figure 7 the source $\mathcal{S}$ of Θ has four elements. Put 1 into a cell if its top element is in $\mathcal{S}$, otherwise put zero. What we get is an $n \times n$ matrix, $\mathbf{M}_L$, which determines L (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form). (The matrix $\mathbf{M}_L$ was originaly mentioned in [18] and first published in [8]) We assume tht the planar diagram is fixed. We fix the WO-direction in the "plain" and the rotation is negative. On possible matrices see [2].

The following matrix is a matrix for N_7:

$$\mathbf{M}_{N_7} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

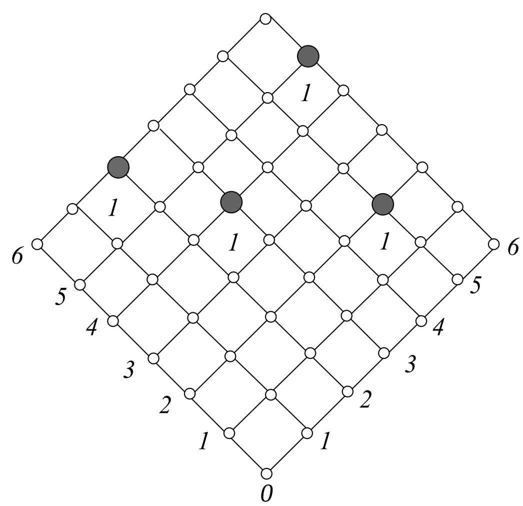


FIGURE 8. A grid and four source elements

Take the following example, given on Figure 8. A source and the corresponding matrix is an $n \times n$ $(0,1)$-matrix, where every row/column contains at most 1 entry, the source elements are $s_1 = (6,2), s_2 = (5,6), s_3 = (4,3), s_4 = (2,5)$:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

For the same lattice there is an another possibility to get a matrix:

$$\begin{vmatrix} 6 & 5 & 4 & 2 \\ 2 & 6 & 3 & 5 \end{vmatrix}$$

3. The proof of Theorem 2

Proof. (1) and (2) are equivalent, see in [4]

(2) $\Rightarrow$ (3). Let L be a 2D semimodular lattice of length n with two dual atoms p and q such that $p \wedge q = 0$, i.e. $0, p, q, 1$ is the skeleton. $\mathbf{J}(L)$ is the set of all x $0 < x \leq p, 0 < x \leq q$. Let $C_1 = \{x; 0 < x \leq p\} \cup \{1\}$ and $C_2 = \{y; 0 < y \leq q\} \cup \{1\}$. $C_1 \approx C_2 \approx \mathcal{C}_{n-1}$. Let's translate to matrices.

$G = C_1 \times C_2 \approx \mathcal{C}_{n-1} \times \mathcal{C}_{n-1}$ is a grid of L (w write the elements of G in the form (x, y), $x, y \in \mathcal{C}_{n-1}$) and the cover-preserving join-homomorphism is $\varphi : (x, y) \Rightarrow x \vee y$. Θ denotes the induced cover-preserving join-congruence of G.

$\langle n, 0\rangle \wedge \langle 0, n\rangle = 0$ and $\langle n, 0\rangle \vee \langle 0, n\rangle = 1$. Let $p = \langle n, 0\rangle$ and $q = \langle 0, n\rangle$. The last row rep. last column of the grid doesn't contain any source element (this would change the order in $\mathbf{J}(G)$). In G/Θ p, q are dual atoms and therefore $\langle n, 1\rangle \equiv \langle n, n\rangle$ (Θ) and $\langle 1, n\rangle \equiv \langle n, n\rangle$ (Θ), which means all other rows/columns must contain a source element, otherwise the meet of p and q would not be 0, i.e. the restricted matrx is invertible.

(3) $\Rightarrow$ (2). Let L be a 2D semimodular lattice and assume that $\mathbf{M}_L$ is a patch matrix. The every row rep. column contains "1" entry, i.e. a source element. If G is the grid then $(n, 1) \equiv (n, n)(\Theta)$, $(1, n) \equiv (n, n)(\Theta)$ but $(n, 1) \not\equiv (n, 0)(\Theta)$, $(n, 1) \not\equiv (0, n)(\Theta)$. In the factor lattice $G/\Theta \cong L$, $p = (n, 0)$, $q = (0, n)$ are dual atoms and $p \wedge q = 0$. $\square$

4. Patch matrices

We consider first $(0, 1)$-matrices, in which every row/column has at most one non zero entry, i.e. "1". A $n \times n$ square matrix $M = [a_{i,j}]$ of this kind is *invertible* if every row/column contains exactly one "1". Obviously, every $(0, 1)$-non-singular matrix is determined by a permutation.

Take a patch matrix, i.e. an $(n + 1) \times (n + 1)$ matrix N, where the last row and the last column contains only zeros and the remaining $n \times n$ matrix is a non-singular matrix M then $N = M_a$ will be called the *augmented* M. If $M = [1]$ then the corresponding augmented matrix is:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 6. *A* block *of a matrix $M = [a_{i,j}]$, $1 \leq i, j \leq n$ is a square submatrix in the form $[a_{i,j}]$, $i \in \{s, s + 1, s + 2, ..., s + k\}$ and $j \in \{t, t + 1, t + 2, ..., t + k\}$ for some s, t, k.*

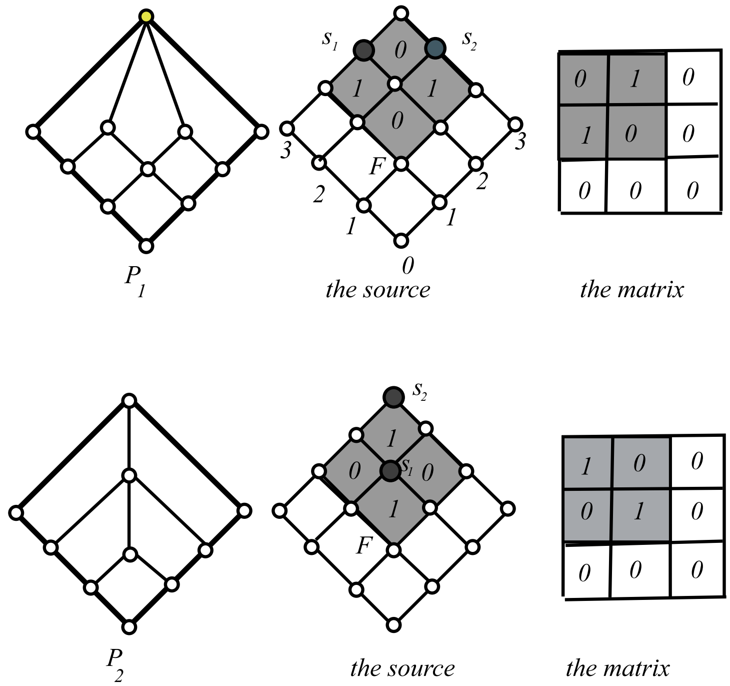


FIGURE 9. Two patch lattices and the matrices

See in Figure 10, the rows/columns of the block are consecutive rows/columns of the given matrix.

Definition 7. *A* block matrix *is a system of blocks of a matrix such that the (set theoretical) meet of two blocks does not contain any entry and every entry is of a block.*

Visually, we have a partition of rectangles (blocks). Let us remark that this definition is not the usual definition. In Figure 10 we have two 4×4-blocks, one 2×2-block and the remaining "0" entries are 1×1-blocks, i.e. *trivial boxes.*

Let M_1 and M_2 two augmented non-singular submatrices as blocks of a matrix N. If $M_1 \cap M_2 \neq \emptyset$, i.e. it contains an entry then there are two possibilities, presented in Figure 11 resp. Figure 12 (the blocks can have different sizes). Then $M_1 \cup M_2$ span a block M. In all other cases we have a row or column with more then one entry "1". These are the vertical and horizontal sum of M_1 and M_2 (see [17]): $M_1 +_v M_2$ resp. $M_1 +_h M_2$ (these are the generated boxes).

We formulate the following easy Theorem (the corresponding theorem to Theorem 1):

Theorem 5. *Every* $(0,1)$*-matrix* M*, in which every row/column has at most one non zero entry, is a block matrix where the blocks are patch matrices and some* 1×1*-matrices (with "0" entries), (i.e. it is the patchwork of patch matrices).*

0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 10. A matrix with three non-trivial blocks

Proof. Let M be a $(0,1)$-matrix in which every row/column has at most one non zero entry. Take the left most 2×2-submatrix M_1 which is an augmented non singular matrices, i.e. has the form given in Figure 10 and 11. If this is a maximal augmented invertible matrix then this is a block. Otherwise, this is not a maximal augmented non singular matrix then there is an other augmented non singular matrix M_2 such that $M_1 \cap M_2 \neq \emptyset$. This implies that $M_1 +_v M_2$ or $M_1 +_v M_2$ exists. These operations are the *nesting of matrices*. On this way we get a maximal augmented non singular matrix. We consider as blocks the maximal augmented non singular $(k \times k)$- matrices. The remaining entries form 1×1 blocks with "0" entries.

Intuitively, we have the "1" entries in the plain, some areas are "density areas" of these entries, these generate a block which is a maximal augmented non singular matrix; the "isolated "1"-s are one-element blocks.

□

Hopefully Theorem 5 allow (planar) semimodular lattices to deal with matrices.

We get in this case:

$$\mathbf{M}_K = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

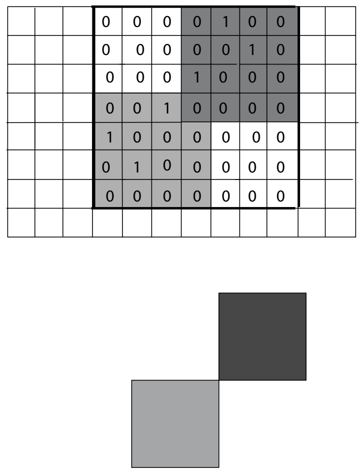
Conjecture 1. *Theorem 5 implies Theorem 2.*

Problem 1. *Establish the connection between Theorem 5 and Theorem 2.*

In Figre 13 you can see a block matrix and the corresponding patchwork.

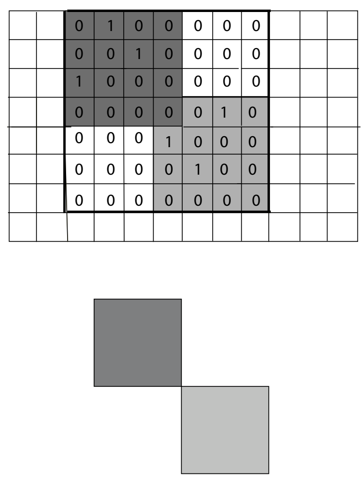
5. Outlook

Theorem 1 is a similar structure theorem of 2D semimodular lattice. I hope similar theorem holds for all semimodular lattices, see more results in [17].



The two restricted matrices

FIGURE 11. Horizontal sum of blocks, $+_h$



The two restricted matrices

FIGURE 12. Vertical sum of blocks, $+_v$

The skeleton, $\mathbf{Sk}(L)$ of a n-dimensional semimodular lattice is a 2^n-element boolean sublattice, which contains $0, 1$. The "**building stones**" of the structure theorem are special rectangular lattices (in most cases the surface of the diagram is a rectangular shape), we get these from Boolean lattices.

The following 3D rectangular lattices are the patch lattices: $\mathcal{C}^3$, M_3, [17]. .

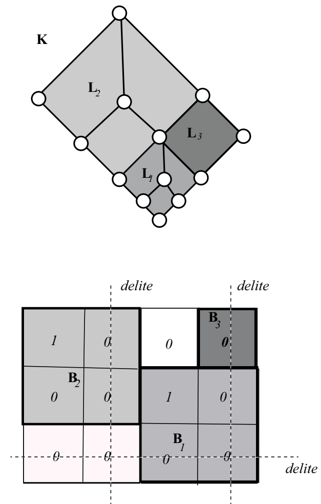


FIGURE 13. The lattice K and the patching of K

Definition 8. *A semimodular lattice is a patchwork lattice if the dual atoms of the skeleton are dual atoms of L.*

Problem 2. *Characterize the patch lattices as nested boolean lattices in the 3D case.*

The direct product $G = C_1 \times C_2 \times C_3$, where C_1, C_2 and C_3 are chains can be considered as a 3D *hypermatrix* (this is a generalization of the matrix to a $n_1 \times n_2 \times n_3$ array of elements: square cuboid), this has a row and two columns. G contains covering cubes, these are called 3-cells. The source elements are top element of the cells, see Figure 8. The 3D hypermatrix of type 2^3 or 3^3 $[a_{i,j,k}]$ is a *source hypermatrix* if $a_{1,1,1} = 1$ and all other entries are zero.

Problem 3. *Characterize the the hypermatrices of patch lattices.*

The "**building tool**" is a kind of gluing, the *patchwork construction*, [17]. It is related to the Hall-Dilworth gluing and S-glued sum (Ch. Herrmann [15]), for instance in the 3-dimensional case we glue together cubes (i.e. 2^3 Boolean lattices) over faces, see in Figure 2. Another example is the Rubik cube, the 27 small cubes ("unit cubes") contact with each other along their sides.

I am convinced that the following conjecture is true and can be traced back to $(0, 1)$-hypermatices.

Conjecture 2. *Every finite semimodular lattice R is the patchwork of patch lattices.*

Some remarks on modular lattices

If R is a rectangular lattice and doesn't contain M_3 (i.e. is diamand free) and we draw the diagram "properly" we get geometricaly a rectangular shape. If R is modular but not distributive then then we get the lattice presented in Figure 14.

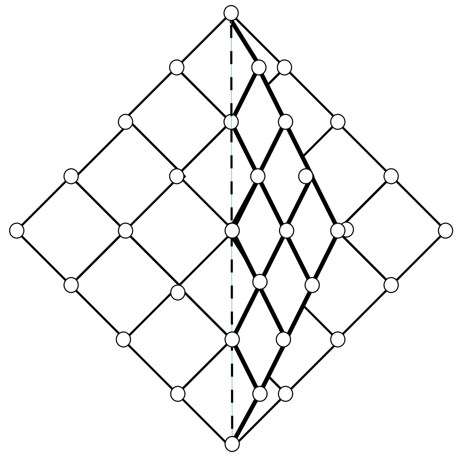


FIGURE 14. $M_3[\mathcal{C}_5]$ a 3D modular rectangular cube, the harlequin

In Figure 14 is the patchwork of the modular lattice $M_3[\mathcal{C}_n]$ ([16]), where the patch lattices (components) are isomorphic to M_3 or ${\mathcal{C}_2}^2$,

You will be surprised to discover that the non modular case in many respects is similar to the distributive case, while the modular case is quite different. In many aspects the diamond, M_3 is the "daredevil" of the theory of semimodular lattices.

Additional results can be found on my website.

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