A]

1) $f(x,y) = y\cos(xy^2)$ $\qquad P\left(\frac{\pi}{2}, 1\right)$

$f'_x(x,y) = y\left(-\sin(xy^2)\right)\cdot y^2 \qquad f'_x\left(\frac{\pi}{2},1\right) = 1\cdot\left(-\sin\frac{\pi}{2}\right)\cdot 1^2$

$= \underline{\underline{-1}}$

$f'_y(x,y) = \cos(xy^2) + y\cdot\left(-\sin(xy^2)\right)\cdot 2xy$

$f'_y\left(\frac{\pi}{2},1\right) = \underbrace{\cos\frac{\pi}{2}}_{\overset{\shortparallel}{0}} + 1\cdot\underbrace{\left(-\sin\frac{\pi}{2}\right)}_{-1}\cdot \cancel{2}\cdot\frac{\pi}{\cancel{2}}\cdot 1 =$

P körül a parc. der. folytonosak, így P-ben tot. deriválható.

$= \underline{\underline{-\pi}}$

Tehát valóban $\operatorname{grad} f\left(\frac{\pi}{2},1\right) = (-1, -\pi)$

Érintősík: $\underline{n} = (-1, -\pi, -1)$ normálvektor

adott pont: $\left(\frac{\pi}{2}, 1, \underbrace{1\cdot\cos\left(\frac{\pi}{2}\cdot 1\right)}_{0}\right)$

$-1\cdot\left(x-\frac{\pi}{2}\right) - \pi(y-1) - 1(z-0) = 0$

$-x + \frac{\pi}{2} - \pi y + \pi - z = 0$

$\underline{\underline{x + \pi y + z = \frac{3}{2}\pi}}$

Iránymenti der.: $\left.\frac{df}{de}\right|_P = -1\cdot\frac{1}{\sqrt{2}} - \pi\cdot\frac{-1}{\sqrt{2}} = \underline{\underline{\frac{\pi-1}{\sqrt{2}}}}$

$\underline{v} = (1, -1)$

A $\operatorname{grad} f = (-1, -\pi)$ irányban max. lesz az iránymenti derivált.

2) $f(x,y) = x\sqrt{x^2+y^2}$

$$f(x,0) = x\cdot|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \Rightarrow f'_x(0,0) = 0$$

$$f(0,y) = 0 \Rightarrow f'_y(0,0) = 0$$

$$\underbrace{f(0+\Delta x, 0+\Delta y) - f(0,0)}_{\Delta x\sqrt{\Delta x^2+\Delta y^2}} = \underbrace{f'_x(0,0)}_{\overset{\shortparallel}{0}}\Delta x + \underbrace{f'_y(0,0)}_{\overset{\shortparallel}{0}}\Delta y + h((\Delta x, \Delta y))$$

$$\Delta x\sqrt{\Delta x^2+\Delta y^2} = h(\Delta x, \Delta y)$$

$$\frac{h(\Delta x, \Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = \Delta x \longrightarrow 0 \quad ((\Delta x, \Delta y) \to (0,0))$$

Azaz tot. deriválható

Ennek kell 0-hoz tartania, hogy tot. der. legyen.

3) $f(x,y) = 4xy + x^4 + y^4$

$f'_x(x,y) = 4y + 4x^3 = 0 \qquad y = -x^3$

$f'_y(x,y) = 4x + 4y^3 = 0 \qquad x = -y^3$

$x = -(-x^3)^3 = x^9$

$x(1-x^8) = 0$

$x = 0 \quad y = 0$

$x = 1 \quad y = -1$

$x = -1 \quad y = 1$

$$H = \begin{bmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{bmatrix}$$

$(0,0)$-ban $\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} < 0$ indef. nincs sz.é.

$(1,-1)$-ben, $(-1,1)$-ben $\begin{vmatrix} 12 & 4 \\ 4 & 12 \end{vmatrix} > 0$ poz. def. lok. min.

$12 > 0$

4) $f(x,y)=xy(12-x-y)$ Belső stac. pontot keresünk:

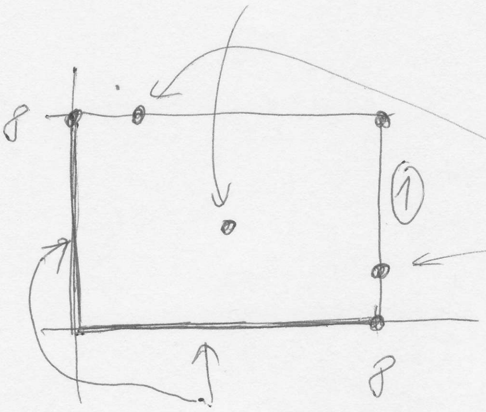
$$\left.\begin{aligned} f'_x(x,y)&=y(12-x-y)+xy(-1)=0\\ f'_y(x,y)&=x(12-x-y)+xy(-1)=0\end{aligned}\right\}$$

$$\left.\begin{aligned} 12-x-y-x&=0\\ 12-x-y-y&=0\end{aligned}\right\} \Rightarrow \left.\begin{aligned} 12-2x-y&=0\\ 12-x-2y&=0\end{aligned}\right\} -$$

$$-x+y=0$$

$$12-3x=0$$

$\underline{\underline{x=4}}$ $\underline{\underline{y=4}}$



(1) $8y(4-y)=32y-8y^2=f(8,y)$

$f'_y(8,y)=32-16y=0$

$y=2$

(2) $8\cancel{x}(4-x)$ $x=2$

konst. 0

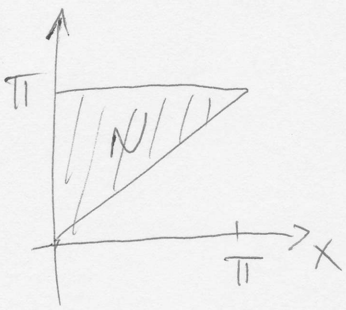
$f(4,4)=4^3$ ← $\underline{\underline{\text{max}}}$

$f(2,8)=32$

$f(8,2)=32$

$f(8,8)=64\cdot(-4)$ ← $\underline{\underline{\text{min}}}$

5)



$$\iint_N \frac{\sin y}{y}\,dx\,dy=\int_0^\pi\int_x^\pi \frac{\sin y}{y}\,dy\,dx=$$

$$=\int_0^\pi\int_0^y \frac{\sin y}{y}\cdot dx\,dy=\int_0^\pi\left[\frac{\sin y}{y}\cdot x\right]_0^y dy=\int_0^\pi \frac{\sin y}{\cancel{y}}\cdot \cancel{y}\,dy$$

$$=\left[-\cos y\right]_0^\pi=-(-1)-(-1)=\underline{\underline{2}}$$

B] $f(x,y) = x\cos(x^2y)$ $\qquad P=(1,\pi/2)$

1) $f'_x(x,y) = \cos(x^2y) + x(-\sin(xy))\cdot 2xy$ $\qquad f'_x(1,\frac{\pi}{2}) =$

$$\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + 1\underbrace{\left(-\sin\frac{\pi}{2}\right)}_{-1}\cdot \cancel{2}\cdot\frac{\pi}{\cancel{2}} = -\pi$$

$$f'_y(x,y) = \underbrace{x}_{1}\cdot\underbrace{(-\sin(x^2\underbrace{y}_{\pi/2}))}_{-1}\cdot\underbrace{x^2}_{1} \qquad f'_y\left(1,\frac{\pi}{2}\right) = -1$$

P körül parc. der. folytonosak, így tot. deriválható. A grad. vektor: $(-\pi,-1)$ valóban.

Érintősík $\quad \underline{n}(-\pi,-1,-1)$

Adott pont $\left(1,\frac{\pi}{2},0\right)$

$$-\pi(x-1) - 1\left(y-\frac{\pi}{2}\right) - 1(z-0) = 0$$

$$-\pi x + \pi - y + \frac{\pi}{2} - z = 0$$

$$\underline{\underline{\pi x + y + z = \frac{3\pi}{2}}}$$

Iránymenti derivált:

$$\left.\frac{df}{d\underline{e}}\right|_P = -\pi\cdot\frac{-1}{\sqrt{2}} + (-1)\cdot\frac{1}{\sqrt{2}} = \underline{\underline{\frac{\pi-1}{\sqrt{2}}}}$$

Max. iránymenti derivált a gradiens irányában lesz. $(-\pi,-1)$ irányban.

2) $f(x,y) = y\sqrt{x^2+y^2}$

$f(x,0) = 0 \quad \Rightarrow \quad f'_x(0,0) = 0$

$f(0,y) = y\cdot|y| = \begin{cases} y^2, & y \geq 0 \\ -y^2, & y < 0 \end{cases} \quad \Rightarrow \quad f'_y(0,0) = 0$

Tot. derivált

$$\underbrace{f(0+\Delta x, 0+\Delta y) - f(0,0)}_{0} = \underbrace{f'_x(0,0)\Delta x}_{0} + \underbrace{f'_y(0,0)}_{0}\Delta y + h(\Delta x, \Delta y)$$

$$\Delta y\sqrt{\Delta x^2+\Delta y^2} = h(\Delta x, \Delta y)$$

Tehát $\dfrac{h(\Delta x, \Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = \Delta y \longrightarrow 0$, ha $(\Delta x, \Delta y) \to (0,0)$

Pont ez kell a tot. deriválhatósághoz.

3) $f(x,y) = 4xy - x^4 - y^4$

$$\left.\begin{aligned} f'_x(x,y) &= 4y - 4x^3 = 0 \\ f'_y(x,y) &= 4x - 4y^3 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= x^9 \\ x(1-x^8) &= 0 \end{aligned}$$

$x = 0 \quad y = 0$

$x = 1 \quad y = 1$ stac. pontok

$x = -1 \quad y = -1$

$$H = \begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$$

$(0,0)$: $\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} < 0$ indef. nincs sz. é.

$(1,1)$, $(-1,-1)$: $\begin{vmatrix} \overset{<0}{-12} & 4 \\ 4 & -12 \end{vmatrix} > 0$ lok. max neg. def.

4) Belső stac. pontot keresünk

$$f(x,y)=xy(18-x-y)$$

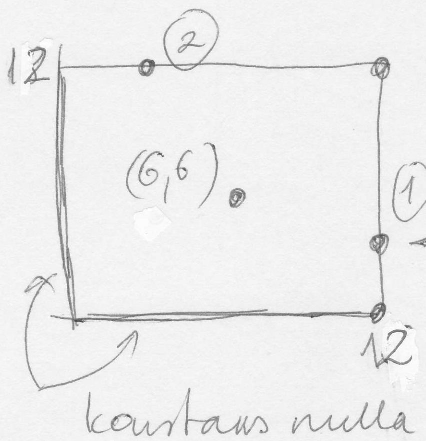
$$\left.\begin{aligned} f'_x(x,y) &= y(18-x-y)+xy(-1)=0 \\ f'_y(x,y) &= x(18-x-y)+xy(-1)=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} 18-2x-y&=0 \\ 18-x-2y&=0 \end{aligned}\right\} -$$

$$-x+y=0 \Rightarrow x=y \Rightarrow 18-3x=0$$

$$x=6 \Rightarrow y=6$$

Peremen



① $f(12,y)=12y(6-y)$

$$(72y-12y^2)'=72-24y=0$$

$$y=3 \quad \text{stac. p.}$$

② $f(x,12)=12x(6-x)$

$$x=3 \quad \text{stac. p.}$$

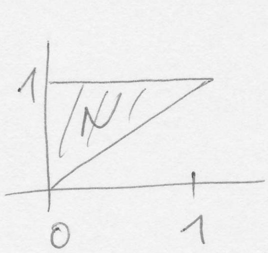
$$f(6,6)=6^3 \leftarrow \underline{\underline{\text{max.}}}$$

$$f(12,3)=36\cdot 3$$

$$f(3,12)=36\cdot 3$$

$$f(12,12)=12^2\cdot(-6) \leftarrow \underline{\underline{\text{min}}}$$

5) $\displaystyle\int_0^1\int_x^1 \frac{e^y-1}{y}\,dy\,dx$



$$\int_0^1\int_0^y \frac{e^y-1}{y}\,dx\,dy = \int_0^1 \frac{e^y-1}{y}\underbrace{[x]_0^y}_{y}\,dy =$$

$$=\int_0^1 (e^y-1)\,dy=\left[e^y-y\right]_0^1 = e-1-(1-0)=\underline{\underline{e-2}}.$$

B3