Határozatlan integrál, primitív függvény

Alapintegrálok

Alapintegráloknak nevezzük az elemi valós függvények differenciálási szabályainak megfordításából adódó primitív függvényeket.

- $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c$, ha $n \neq -1$; mert $\left(\dfrac{x^{n+1}}{n+1} + c\right)' = x^n$
- $\int \dfrac{1}{x} \, dx = \ln|x| + c$; mert $(\ln + c)' = \dfrac{1}{x}$
- $\int \sin x \, dx = -\cos x + c$; mert $(-\cos x + c)' = \sin x$
- $\int \cos x \, dx = \sin x + c$; mert $(\sin x + c)' = \cos x$
- $\int \dfrac{1}{\cos^2 x} \, dx = \tan x + c$; mert $(\tan x + c)' = \dfrac{1}{\cos^2 x}$
- $\int \dfrac{1}{\sin^2 x} \, dx = -\cot x + c$; mert $(-\cot x + c)' = \dfrac{1}{\sin^2 x}$
- $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$; mert $(\arcsin x + c)' = \dfrac{1}{\sqrt{1-x^2}}$
- $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = -\arccos x + c$; mert $(-\arccos x + c)' = \dfrac{1}{\sqrt{1-x^2}}$
- $\int \dfrac{1}{1+x^2} \, dx = \arctan x + c$; mert $(\arctan x + c)' = \dfrac{1}{1+x^2}$
- $\int \dfrac{1}{1+x^2} \, dx = -\mathrm{arccot}\, x + c$; mert $(-\mathrm{arccot}\, x + c)' = \dfrac{1}{1+x^2}$
- $\int \cosh x \, dx = \sinh x + c$; mert $(\sinh x + c)' = \cosh x$
- $\int \sinh x \, dx = \cosh x + c$; mert $(\cosh x + c)' = \sinh x$
- $\int \dfrac{1}{\cosh^2 x} \, dx = \tanh x + c$; mert $(\tanh x + c)' = \dfrac{1}{\cosh^2 x}$
- $\int \dfrac{1}{\sinh^2 x} \, dx = -\coth x + c$; mert $(-\coth x + c)' = \dfrac{1}{\sinh^2 x}$
- $\int \mathrm{e}^x \, dx = \mathrm{e}^x + c$; mert $(\mathrm{e}^x + c)' = \mathrm{e}^x$

- $\int a^x \, dx = \frac{a^x}{\ln a} + c;$ mert $\left(\frac{a^x}{\ln a} + c\right)' = a^x$
- $\int \frac{1}{\sqrt{x^2+1}} \, dx = \operatorname{ar\,sinh} x + c;$ mert $(\operatorname{ar\,sinh} x + c)' = \frac{1}{\sqrt{x^2+1}}$
- $\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{ar\,cosh} x + c;$ mert $(\operatorname{ar\,cosh} x + c)' = \frac{1}{\sqrt{x^2-1}}$
- $\int \frac{1}{1-x^2} \, dx = \operatorname{ar\,tanh} x + c$, ha $|x| < 1$; mert $(\operatorname{ar\,tanh} x + c)' = \frac{1}{1-x^2}$
- $\int \frac{1}{1-x^2} \, dx = \operatorname{ar\,coth} x + c$, ha $|x| > 1$; mert $(\operatorname{ar\,coth} x + c)' = \frac{1}{1-x^2}$

Az integrálás alapképleteinek és -szabályainak alkalmazása

Példák:

(A/1) $\int \frac{dx}{x^2} = \int x^{-2} \, dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + c$

(A/2) $\int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}} \, dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} = \frac{3}{2}\sqrt[3]{x^2} + c$

(A/3) $\int x^2(x^2-1) \, dx = \int (x^4 - x^2) \, dx = \int x^4 \, dx - \int x^2 \, dx = \frac{x^5}{5} - \frac{x^3}{3} + c$

(A/4) $\int (x^2-1)^2 \, dx = \int (x^4 - 2x^2 + 1) \, dx = \int x^4 \, dx - 2\int x^2 \, dx + \int 1 \, dx =$
$= \frac{x^5}{5} - 2\frac{x^3}{3} + x + c$

(A/5) $\int \frac{\sqrt{x} - x + x^4}{x^2} \, dx = \int (x^{-\frac{3}{2}} - \frac{1}{x} + x^2) \, dx = -\frac{2}{\sqrt{x}} - \ln|x| + \frac{x^3}{3} + c$

(A/6) $\int \frac{(x+1)^2}{\sqrt{x}} \, dx = \int (x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}) \, dx = \frac{2}{5}\sqrt{x^5} + \frac{4}{3}\sqrt{x^2} + 2\sqrt{x} + c$

(A/7) $\int \frac{x^2 - 4x + 7}{x-2} \, dx = \int \frac{(x-2)^2 + 3}{x-2} \, dx = \int (x-2) \, dx + \int \frac{3}{x-2} \, dx =$
$= \frac{x^2}{2} - 2x + 3\ln|x-2| + c$

(A/8) $\int \frac{1+2x^2}{x^2(1+x^2)} \, dx = \int \left(\frac{1+x^2}{x^2(1+x^2)} + \frac{x^2}{x^2(1+x^2)} \right) dx =$
$\int \left(\frac{1}{x^2} + \frac{1}{1+x^2} \right) dx = -\frac{1}{x} + \arctan x + c$

(A/9) $\displaystyle\int \frac{6}{5+5x^2}\,dx = \frac{6}{5}\int \frac{1}{1+x^2}\,dx = \frac{6}{5}\arctan x + c$

(A/10) $\displaystyle\int \frac{\ln 2}{\sqrt{2+2x^2}}\,dx = \frac{\ln 2}{\sqrt{2}}\int \frac{1}{\sqrt{1+x^2}}\,dx = \frac{\ln 2}{\sqrt{2}}\,\mathrm{ar\,sinh} x + c$

(A/11) $\displaystyle\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx =$

$\displaystyle\int \left(\frac{1}{\cos^2 x} - 1\right)dx = \tan x - x + c$

(A/12) $\displaystyle\int \frac{\cos 2x}{\cos x - \sin x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}\,dx =$

$\displaystyle\int (\cos x + \sin x)\,dx = \sin x - \cos x + c$

(A/13) $\displaystyle\int \frac{x^4}{1-x}\,dx = -\int \left(\frac{x^4-1}{x-1} + \frac{1}{x-1}\right)dx =$

$\displaystyle-\int \left(x^3 + x^2 + x + 1 + \frac{1}{1-x}\right)dx =$

$\displaystyle-\left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1|\right) + c$

Feladatok:

(a/1) $\displaystyle\int (\sqrt{x}+1)(x-\sqrt{x}+1)\,dx$

(a/2) $\displaystyle\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\,dx$

Integrálás helyettesítéssel

$$\boxed{\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt; \quad \text{helyettesítés: } x = \varphi(t)}$$

Példák:

(B/1) $\displaystyle\int \cos(4x-5)\,dx = \frac{1}{4}\int \cos t\,dt = \frac{1}{4}\sin t + c = \frac{1}{4}\sin(4x-5) + c$

helyettesítés: $t = 4x - 5 \Longrightarrow \dfrac{dt}{dx} = 4 \Longrightarrow dx = \dfrac{1}{4}\,dt$

(B/2) $\int \sqrt{8-2x}\; dx = -\frac{1}{2}\int \sqrt{t}\; dt = \frac{1}{2}\,\frac{2}{3}\,t^{\frac{3}{2}} + c = -\frac{1}{3}\sqrt{(8-2x)^3} + c$

helyettesítés: $t = 8-2x \Longrightarrow \frac{dt}{dx} = -2 \Longrightarrow dx = -\frac{1}{2}\, dt$

(B/3) $\int \mathrm{e}^{-x}\; dx = -\int \mathrm{e}^{t}\; dt = -\mathrm{e}^{u} + c = -\mathrm{e}{-}x + c$

helyettesítés: $t = -x \Longrightarrow \frac{dt}{dx} = -1 \Longrightarrow dx = -\; dt$

(B/4) $\int 10^x \mathrm{e}^x\; dx = \mathrm{e}^{\ln 10}\mathrm{e}^x = \int \mathrm{e}^{x(\ln 10+1)}\; dx = \frac{1}{\ln 10+1}\mathrm{e}^t + c = \frac{\mathrm{e}^{x(\ln 10+1)}}{\ln 10+1} + c$

helyettesítés: $t = x(\ln 10+1) \Longrightarrow \frac{dt}{dx} = \ln 10 + 1 \Longrightarrow dx = \frac{1}{\ln 10+1}\; dt$

(B/5) $\int \frac{1}{5+x^2}\; dx = \frac{1}{5}\int \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}\; dx = \frac{1}{5}\sqrt{5}\arctan\frac{x}{\sqrt{5}} + c$

helyettesítés: $t = \frac{x}{\sqrt{5}} \Longrightarrow \frac{dt}{dx} = \frac{1}{\sqrt{5}} \Longrightarrow dx = \sqrt{5}\, dt$

(B/6) $\int \frac{1}{(2x-3)^5}\; dx = \int (2x-3)^{-5}\; dx = \frac{1}{2}\int t^{-5}\; dt =$

$\frac{1}{2}\frac{t^{-4}}{-4} + c = -\frac{1}{8}\frac{1}{(2x-3)^4} + c$

helyettesítés: $t = 2x-3 \Longrightarrow \frac{dt}{dx} = 2 \Longrightarrow dx = \frac{1}{2}\; dt$

(B/7) $\int x^2\sqrt[3]{x^3+8}\; dx = \frac{1}{3}\int 3x^2\sqrt[3]{x^3+8}\; dx = \frac{1}{3}\int 3\sqrt[3]{t}\; dt$

$= \frac{1}{3}\,\frac{3}{4}\,t^{\frac{4}{3}} + c = \frac{1}{4}\sqrt[3]{(x^3+8)^4} + c$

helyettesítés: $t = x^3+8 \Longrightarrow \frac{dt}{dx} = 3x^2 \Longrightarrow dt = 3x^2\; dx$

(B/8) $\int x\sin(x^2+2)\; dx = \frac{1}{2}\int 2x\sin(x^2+2)\; dx = \frac{1}{2}\int \sin t\; dt =$

$-\frac{1}{2}\cos t + c = -\frac{1}{2}\cos(x^2+2) + c$

helyettesítés: $t = x^2+2 \Longrightarrow \frac{dt}{dx} = 2x \Longrightarrow dt = 2x\; dx$

(B/9) $\int \frac{\sqrt[3]{\tan x}}{\cos^2 x}\; dx = \int \sqrt[3]{t}\; dt = \frac{3}{4}t^{\frac{4}{3}} + c = \frac{3}{4}\sqrt[3]{\tan^4 x} + c$

helyettesítés: $t = \tan x \Longrightarrow \dfrac{dt}{dx} = \dfrac{1}{\cos^2 x} \Longrightarrow dt = \dfrac{1}{\cos^2 x}\, dx$

(B/10) $\displaystyle\int \frac{x}{x^4+1}\, dx = \frac{1}{2}\int \frac{1}{t^2+1}\, dt = \frac{1}{2}\arctan t + c = \frac{1}{2}\arctan x^2 + c$

helyettesítés: $t = x^2 \Longrightarrow \dfrac{dt}{dx} = 2x \Longrightarrow dt = 2x\, dx$

(B/11) $\displaystyle\int \frac{x^3}{\sqrt{x^8-1}}\, dx = \frac{1}{4}\int \frac{1}{\sqrt{t^2-1}}\, dt = \frac{1}{4}\,\mathrm{ar\,cosh} t + c = \frac{1}{4}\,\mathrm{ar\,cosh} x^4 + c$

helyettesítés: $t = x^4 \Longrightarrow \dfrac{dt}{dx} = 4x^3 \Longrightarrow dt = 4x^3\, dx$

Feladatok:

(b/1) $\displaystyle\int \sin\left(\frac{\pi}{3} - 3x\right) dx$

(b/2) $\displaystyle\int \frac{3}{\sqrt{3x^2-2}}\, dx$

(b/3) $\displaystyle\int \sqrt[5]{(8-3x)^6}\, dx$

(b/4) $\displaystyle\int x\sqrt{1-x^2}\, dx$

(b/5) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\, dx$

(b/6) $\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\, dx$

(b/7) $\displaystyle\int \frac{\sqrt{\ln x}}{x}\, dx$

(b/8) $\displaystyle\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx$

Integrálás az $\boxed{\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c}$ szabállyal

Példák:

(C/1) $\displaystyle\int \frac{x}{4+x^2}\, dx = \frac{1}{2}\int \frac{2x}{4+x^2}\, dx = \ln(4+x^2) + c$

(lehet $t = 4 + x^4$ helyettesítéssel is)

(C/2) $\displaystyle\int \frac{1}{x \ln x}\, dx = \int \frac{\frac{1}{x}}{\ln x}\, dx = \ln \ln x + c$

(lehet $t = \ln x$ helyettesítéssel is)

(C/3) $\displaystyle\int \frac{x+2}{2x-1}\, dx = \int \left(\frac{x - \frac{1}{2}}{2x-1} + \frac{\frac{5}{2}}{2x-1} \right) dx =$

$\displaystyle\int \left(\frac{1}{2} + \frac{5}{4} \frac{2}{2x-1} \right) dx = \frac{x}{2} + \frac{5}{4} \ln |2x-1| + c$

(lehet $t = 2x - 1$ helyettesítéssel is)

(C/4) $\displaystyle\int \frac{3x-1}{x^2+9}\, dx = \int \frac{3x}{x^2+9}\, dx - \int \frac{1}{x^2+9}\, dx =$

$\displaystyle\frac{3}{2} \int \frac{2x}{x^2+9}\, dx - \frac{1}{9} \int \frac{1}{1 + \left(\frac{x}{3}\right)^2}\, dx \overset{*}{=} \frac{3}{2} \ln (x^2+9) - \frac{1}{3} \int \frac{1}{1+t^2}\, dx =$

$\displaystyle= \frac{3}{2} \ln (x^2+9) - \frac{1}{3} \arctan t + c = \frac{3}{2} \ln (x^2+9) - \frac{1}{3} \arctan \frac{x}{3} + c$

$*$ helyettesítés a második integrálban: $t = \dfrac{x}{3} \Longrightarrow \dfrac{dt}{dx} = \dfrac{1}{3} \Longrightarrow dt = \dfrac{1}{3}\, dx$

Parciális integrálás

$$\boxed{\int u(x) v'(x)\, dx = u(x) v(x) - \int u'(x) v(x)\, dx}$$

Példák:

(D/1) $\displaystyle\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c$

$u = x \qquad u' = 1$

$v' = \cos x \qquad v = \sin x$

(D/2) $\displaystyle\int (x^2 - 1) \sin 3x\, dx \overset{(1)}{=} -\frac{1}{3}(x^2-1)\cos 3x + \frac{2}{3} \int x \cos 3x\, dx \overset{(2)}{=}$

(1) $u = x^2 - 1 \qquad u' = 2x$

$v' = \sin 3x \qquad v = -\dfrac{1}{3} \cos 3x$

(2) $u = x \qquad u' = 1$

$v' = \cos 3x \qquad v = \dfrac{1}{3} \sin 3x$

$$-\frac{1}{3}(x^2-1)\cos 3x + \frac{2}{3}\left(\frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\ dx\right) =$$

$$-\frac{1}{3}(x^2-1)\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + c$$

(D/3) $\displaystyle\int \sin\sqrt{x}\ dx \overset{(1)}{=} 2\int t\sin t\ dt \overset{(2)}{=}$

(1) helyettesítés: $x = t^2 \Longrightarrow dx = 2t\ dt$

(2) $u = t$ $\quad u' = 1$
$v' = \sin t$ $\quad v = -\cos t$

$$-2t\cos t + 2\int \cos t\ dt = -2t\cos t + 2\sin t = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + c$$

(D/4) $\displaystyle\int x\sin x\cos x\ dx = \frac{1}{2}\int x\sin 2x\ dx =$

$u = x$ $\quad u' = 1$
$v' = \sin 2x$ $\quad v = -\frac{1}{2}\cos 2x$

$$-\tfrac{1}{4}x\cos 2x + \tfrac{1}{4}\int \cos 2x\ dx = -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x + c$$

(D/5) $\displaystyle\int x\arctan x\ dx =$

$u = \arctan x$ $\quad u' = \dfrac{1}{1+x^2}$
$v' = x$ $\quad v = \dfrac{x^2}{2}$

$$\frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\ dx =$$
$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + c$$

(D/6) $\displaystyle\int e^{3x}\cos 2x\ dx \overset{(1)}{=} \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x\ dx \quad (*)$

$$\begin{array}{lll} (1) & u = \mathrm{e}^{3x} & u' = 3\mathrm{e}^{3x} \\ & v' = \cos 2x & v = \dfrac{1}{2}\sin 2x \end{array}$$

Másrészt:

$$\int \mathrm{e}^{3x}\cos 2x\,dx \overset{(2)}{=} \frac{1}{3}\mathrm{e}^{3x}\cos 2x + \frac{2}{3}\int \mathrm{e}^{3x}\sin 2x\,dx \qquad (**)$$

$$\begin{array}{lll} (2) & u = \cos 2x & u' = -2\sin 2x \\ & v' = \mathrm{e}^{3x} & v = \dfrac{1}{3}\mathrm{e}^{3x} \end{array}$$

A (*) egyenletet 4-gyel, a (**) egyenletet 9-cel szorozva és összeadva a $\int \mathrm{e}^{3x}\sin 2x\,dx$ tag kiesik, így kapjuk:

$$13\int \mathrm{e}^{3x}\cos 2x\,dx = 2\mathrm{e}^{3x}\sin 2x + 3\mathrm{e}^{3x}\cos 2x$$

Tehát:

$$\int \mathrm{e}^{3x}\cos 2x\,dx = \frac{2\mathrm{e}^{3x}\sin 2x + 3\mathrm{e}^{3x}\cos 2x}{13} + c$$

(D/7) $\displaystyle\int \mathrm{e}^{\arcsin x}\,dx \overset{*}{=} \int \mathrm{e}^{t}\cos t\,dt = ?$

$*$ helyettesítés: $t = \arcsin x \Longrightarrow x = \sin t \Longrightarrow dx = \cos t dt$

$$\int \mathrm{e}^{t}\cos t\,dt \overset{(1)}{=} \mathrm{e}^{t}\sin t - \int \mathrm{e}^{t}\sin t\,dt$$

$$\begin{array}{lll} (1) & u = \mathrm{e}^{t} & u' = \mathrm{e}^{t} \\ & v' = \cos t & v = \sin t \end{array}$$

Másrészt:

$$\int \mathrm{e}^{t}\cos t\,dt \overset{(2)}{=} \mathrm{e}^{t}\cos t + \int \mathrm{e}^{t}\sin t\,dt$$

$$\begin{array}{lll} (2) & u = \cos t & u' = -\sin t \\ & v' = \mathrm{e}^{t} & v = \mathrm{e}^{t} \end{array}$$

A két egyenletetet összadva az $\int \mathrm{e}^{t}\sin t\,dt$ tag kiesik, így:

$$2\int \mathrm{e}^{t}\cos t\,dt = \mathrm{e}^{t}\sin t + \mathrm{e}^{t}\cos t = \mathrm{e}^{t}(\sin t + \cos t)$$

Tehát:

$$\int e^{\arcsin x}\ dx = \frac{e^{\arcsin x}}{2}(\overbrace{\sin \arcsin x}^{x} + \overbrace{\cos \arcsin x}^{\sqrt{1-x^2}}) =$$

$$\frac{e^{\arcsin x}}{2}(x + \sqrt{1-x^2}) + c$$

Feladatok:

(d/1) $\displaystyle\int \left(\frac{x+2}{e^x}\right)^2\ dx$

(d/2) $\displaystyle\int x^2\ a^x\ dx$

(d/3) $\displaystyle\int x^3\ e^{-x^2}\ dx$

(d/4) $\displaystyle\int \arctan \sqrt{x}\ dx$

(d/5) $\displaystyle\int \ln^3 x\ dx$ (segítség: $u = \ln^3 x$, $v' = 1$; majd további két ehhez hasonló parc. int.)

(d/6) $\displaystyle\int xe^x dx$

(d/7) $\displaystyle\int (8x^2 - 11x + 5)e^x dx$

(d/8) $\displaystyle\int x \arcsin x dx$

(d/9) $\displaystyle\int x^3 3^x dx$

(d/10) $\displaystyle\int \sin \sqrt{x} dx$

(d/11) $\displaystyle\int x \mathrm{arctg} x dx$

(d/12) $\displaystyle\int x^2 \cos 2x dx$

(d/13) $\displaystyle\int x \sin x \cos x dx$

(d/14) $\int \ln x dx$

(d/15) $\int \ln^2 x dx$

(d/16) $\int \ln^3 x dx$

(d/17) $\int (\arcsin x)^2 dx$

(d/18) $\int e^x \cos x dx$

(d/19) $\int 2^x \sin x dx$

(d/20) $\int e^{\arcsin x} dx$

(d/21) $\int e^{3x} \cos 2x dx$

(d/22) $\int \frac{x+2^{2}}{e^x} dx$

(d/23) $\int x \ln x dx$

Racionális törtfüggvények integrálása

Példák:

(E/1) $\int \frac{x-2}{x^2-7x+12} dx = ?$

Az integrandust parciális törtekre bontjuk:

$$\frac{x-2}{x^2-7x+12} = \frac{x-2}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} =$$

$$= \frac{A(x-4)+B(x-3)}{(x-3)(x-4)} = \frac{x(A+B)-4A-3B}{(x-3)(x-4)}$$

Összehasonlítva a számlálóban az együtthatókat a jobb és bal oldalon, a következő egyenletrendszert kapjuk:

$$A + B = 1$$
$$-4A - 3B = -2$$

melynek megoldása: $A = -1$, $B = 2$. Tehát:

$$\int \frac{x-2}{x^2 - 7x + 12}\, dx = \int \left(-\frac{1}{x-3} + \frac{2}{x-4} \right) dx = -\int \frac{1}{x-3}\, dx + 2\int \frac{1}{x-4}\, dx =$$
$$= -\ln|x-3| + 2\ln|x-4| + c = \ln c\frac{(x-4)^2}{|x-3|}$$

(E/2) $\displaystyle\int \frac{x}{x^4 - 3x^2 + 2}\, dx \overset{*}{=} \frac{1}{2}\int \frac{1}{t^2 - 3t + 2}\, dt = ?$

$*$ helyettesítés: $t = x^2 \Longrightarrow dt = 2xdx$

Az integrandust parciális törtekre bontjuk:

$$\frac{1}{t^2 - 3t + 2} = \frac{1}{(t-1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2} =$$
$$= \frac{A(t-2) + B(t-1)}{(t-1)(t-2)} = \frac{t(A+B) - 2A - B}{(t-1)(t-2)}$$

Összehasonlítva a számlálóban az együtthatókat, a következő egyenletrendszert kapjuk:

$$A + B = 0$$
$$-2A - B = 1$$

melynek megoldása: $A = -1$, $B = 1$. Tehát:

$$\frac{1}{2}\int \frac{1}{t^2 - 3t + 2}\, dx = \frac{1}{2}\int \left(-\frac{1}{t-1} + \frac{1}{t-2} \right) dx =$$
$$= -\frac{1}{2}\int \frac{1}{t-1}\, dx + \frac{1}{2}\int \frac{1}{t-2}\, dx =$$
$$= -\frac{1}{2}\ln|t-1| + \frac{1}{2}\ln|t-2| + \frac{1}{2}\ln c =$$
$$= \frac{1}{2}\ln c\frac{|t-2|}{|t-1|} = \frac{1}{2}\ln c\frac{|x^2-2|}{|x^2-1|}$$

(E/3) $\displaystyle\int \frac{3x-2}{x^2+4x+8}\,dx = ?$

Itt a nevező nem bontható fel lineáris tényezők szorzatára, ezért az integrandust két olyan tört összegére bontjuk, amelyek egyikében a a számláló a nevező deriváltjának konstansszorosa, a másik számlálója pedig konstans:

$$\frac{3x-2}{x^2+4x+8} = \frac{\alpha(2x+4)}{x^2+4x+8} + \frac{\beta}{x^2+4x+8} = \frac{2\alpha x+4\alpha+\beta}{x^2+4x+8}$$

Összehasonlítva a számlálóban az együtthatókat, a következő egyenletrendszert kapjuk:

$$\begin{aligned} 2\alpha &= 3 \\ 4\alpha+\beta &= -2 \end{aligned}$$

melynek megoldása: $\alpha = \dfrac{3}{2}$, $\beta = -8$. Tehát:

$$\begin{aligned} \int \frac{3x-2}{x^2+4x+8}\,dx &= \frac{3}{2}\int \frac{2x+4}{x^2+4x+8}\,dx - 8\int \frac{1}{x^2+4x+8}\,dx = \\ &= \frac{3}{2}\ln\left(x^2+4x+8\right) - 8\int \frac{1}{4}\,\frac{1}{(\frac{x+2}{2})^2+1}\,dx = \\ &= \frac{3}{2}\ln\left(x^2+4x+8\right) - 4\arctan\frac{x+2}{2} + c \end{aligned}$$

(E/4) $\displaystyle\int \frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx = ?$

Az integrandust parciális törtekre bontjuk:

$$\begin{aligned} \frac{x^3-2x^2+4}{x^3(x-2)^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^2} + \frac{D}{(x-2)} + \frac{E}{(x-2)^2} = \\ &= \frac{A[x^2(x-2)^2] + B[x(x-2)^2] + C[(x-2)^2] + D[x^3(x-2)] + E[x^3]}{x^3(x-2)^2} \end{aligned}$$

Összehasonlítva a számlálóban az együtthatókat, a következő egyenletrendszert kapjuk:

$$\begin{aligned}
A + D &= 0 \\
-4A + B - 2D + E &= 1 \\
4A - 4B + C &= -2 \\
4B - 4C &= 0 \\
4C &= 4
\end{aligned}$$

melynek megoldása: $A = \frac{1}{4}$, $B = 1$, $C = 1$, $D = -\frac{1}{4}$, $E = \frac{1}{2}$. Tehát:

$$\begin{aligned}
\int \frac{x^3 - 2x^2 + 4}{x^3(x-2)^2}\, dx &= \int \left(\frac{1}{4x} + \frac{1}{x^2} + \frac{1}{x^2} - \frac{1}{4(x-2)} + \frac{1}{2(x-2)^2} \right) dx == \\
&= \frac{1}{4} \ln \frac{x}{x-2} - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{2(x-1)} + c
\end{aligned}$$

(E/5) $\displaystyle\int \frac{1}{x^6 + x^4}\, dx = \int \frac{1}{x^4(x^2+1)}\, dx = ?$

Az integrandust parciális törtekre bontjuk:

$$\begin{aligned}
\frac{1}{x^4(x^2+1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{Ex + F}{x^2 + 1} = \\
&= \frac{A(x^5 + x^3) + B(x^4 + x^2) + C(x^3 + x) + D(x^2 + 1) + Ex^5 + Fx^4}{x^4(x^2+1)}
\end{aligned}$$

Összehasonlítva a számlálóban az együtthatókat, a következő egyenletrendszert kapjuk:

$$\begin{aligned}
A + E &= 0 \\
B + F &= 0 \\
A + C &= 0 \\
B + D &= 0 \\
C &= 0 \\
D &= 1
\end{aligned}$$

melynek megoldása: $A = 0$, $B = -1$, $C = 0$, $D = 1$, $E = 0$, $F = 1$. Tehát:

$$\int \frac{1}{x^6 + x^4}\, dx = \int \left(-\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^2+1} \right) dx = \frac{1}{x} - \frac{1}{3x^2} + \arctan x + c$$

Feladatok:

(e/1) $\displaystyle\int \frac{2x^2-5}{x^4-5x^2+6}\,dx$

(e/2) $\displaystyle\int \frac{1}{x^4-x^2}\,dx$

(e/3) $\displaystyle\int \frac{x}{x^3-1}\,dx$

(e/4) $\displaystyle\int \frac{x^2}{1-x^4}\,dx$

(e/5) $\displaystyle\int \frac{1}{(x+1)^2(x^2+1)}\,dx$

(e/6) $\displaystyle\int \frac{2}{5x-1}dx$

(e/7) $\displaystyle\int \frac{\pi}{x+2}dx$

(e/8) $\displaystyle\int \frac{2x+1}{3x-4}dx$

(e/9) $\displaystyle\int \frac{7x-2}{4x+11}dx$

(e/10) $\displaystyle\int \frac{x^2+1}{x+1}dx$

(e/11) $\displaystyle\int \frac{x^2-2x+3}{5x+7}dx$

(e/12) $\int \frac{3x^3+2x^2+x}{10x-1}dx$

(e/13) $\displaystyle\int \frac{17dx}{(3x+14)^2}$

(e/14) $\displaystyle\int \frac{33dx}{(x+19)^3}$

(e/15) $\displaystyle\int \frac{x+7}{(x+2)^2}dx$

(e/16) $\displaystyle\int \frac{x^2-x+8}{(x-3)^3}dx$

(e/17) $\displaystyle\int \frac{x^3+x^2-x+1}{(x-1)^4}dx$

(e/18) $\displaystyle\int \frac{x^4+1}{(x-2)^3}dx$

(e/19) $\displaystyle\int \frac{x^7}{(1+x)^6}dx$

(e/20) $\displaystyle\int \frac{dx}{3x^2+4x+5}$

(e/21) $\displaystyle\int \frac{dx}{5x^2-4x+3}$

(e/22) $\displaystyle\int \frac{dx}{x^2-6x-10}$

(e/23) $\displaystyle\int \frac{x-4}{x^2+4}dx$

(e/24) $\displaystyle\int \frac{7x-6}{x^2-8}$

(e/25) $\displaystyle\int \frac{3x^4+4x^3-x+13}{x^2+3x-4}dx$

(e/26) $\displaystyle\int \frac{x^6-x^5}{x^2-5x-6}dx$

(e/27) $\displaystyle\int \frac{x^4+x^2+1}{x^3+3x^2+3x+1}dx$

(e/28) $\displaystyle\int \frac{x+3}{x^3+4x^2+4x}dx$

(e/29) $\displaystyle\int \frac{(x^2+1)dx}{(x^2-10x+21)(x^2+2x+1)}$

(e/30) $\displaystyle\int \frac{x^4+x^2+1}{(x+1)^2(x-1)^2}dx$

(e/31) $\displaystyle\int \frac{dx}{(x^2+1)^2}$

(e/32) $\displaystyle\int \frac{dx}{(x^2+1)^3}$

(e/33) $\displaystyle\int \frac{3x-4}{(x^2+4)^3}$

(e/34) $\displaystyle\int \frac{x+1}{x^4+4x^2+3}dx$

(e/35) $\displaystyle\int \frac{x^3+x+2}{(x^2+x+1)(x-2)}dx$

(e/36) $\displaystyle\int \frac{x^3+2}{x^4-4x^2+3}dx$

(e/37) $\displaystyle\int \frac{xdx}{x^4-2x^2-3}$

(e/38) $\displaystyle\int \frac{xdx}{x^4-3x^2+2}$

Trigonometrikus és hiperbolikus függvények integrálása

Páratlan kitevő esetén: leválasztás + helyettesítés.
Páros kitevő esetén: linearizálás.
Linearizálási formulák:

$$\begin{array}{lll}
\cos^2 x+\sin^2 x=1; & \cos 2x=\cos^2 x-\sin^2 x; & \sin 2x = 2\sin x\cos x; \\
\cos^2 x = 1-\sin^2 x; & \sin^2 x = 1-\cos^2 x; & \\
\cos^2 x=\dfrac{1+\cos 2x}{2}; & \sin^2 x=\dfrac{1-\cos 2x}{2}; & \\
\cosh^2 x-\sinh^2 x=1; & \cosh 2x=\cosh^2 x+\sinh^2 x; & \sinh 2x=2\sinh x\cosh x; \\
\cosh^2 x=1+\sinh^2 x; & \sinh^2 x=\cosh^2 x-1; & \\
\cosh^2 x=\dfrac{\cosh 2x+1}{2}; & \sinh^2 x=\dfrac{\cosh 2x-1}{2} &
\end{array}$$

Példák:

(F/1) $\displaystyle\int \cos^5 x\;dx=\int \cos^4 x\cos x\;dx=\int(\cos^2 x)^2\cos x\;dx=$

$$\int(1-\sin^2 x)^2\cos x\;dx\overset{*}{=}\int(1-t^2)^2\;dt=\int(1-2t^2+t^4)\;dt=$$

$$t-\frac{2t^3}{3}+\frac{t^5}{5}+c=\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c$$

($*$ helyettesítés: $t=\sin x\Longrightarrow dt=\cos x dx$)

(F/2) $\int \sin^6 x\, dx = \int (\sin^2 x)^3\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^3\, dx =$

$= \frac{1}{8}\int \left(1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\right)\, dx =$

$= \frac{1}{8}\int \left(1 - 3\cos 2x + 3\frac{1+\cos 4x}{2} - \cos^3 2x\right)\, dx =$

$= \cdots$ (az utolsó tagra a páratlan kitevő módszerét alkalmazva) $\cdots =$

$= \frac{1}{8}\left(\frac{5}{2}x - \frac{3}{2}\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{2}\sin 2x + \frac{1}{6}\sin^3 2x\right) + c$

(F/3) $\int \sinh^2 x \cosh^3 x\, dx = \int \sinh^2 x(1+\sinh^2 x)\overbrace{\cosh x\, dx}^{dt} = \int t^2(1+t^2)\, dt =$

$= \frac{t^3}{3} + \frac{t^5}{5} = \frac{\sinh^3 x}{3} + \frac{\sinh^5 x}{5} + c$

(F/4) $\int \frac{\sinh^3 x}{\sqrt{\cosh x}}\, dx = \int \frac{(\cosh^2 x - 1)\sinh x}{\sqrt{\cosh x}}\, dx = \int \frac{t^2-1}{\sqrt{t}}\, dt = \frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}} =$

$=$

$\frac{2}{5}\sqrt{\cosh^5 x} - \frac{2}{3}\sqrt{\cosh^3 x} + c$

Feladatok:

(f/1) $\int \frac{\sin^4 x}{\cos^2 x}\, dx$

(f/2) $\int \sin^6 x \cos^3 x\, dx$

(f/3) $\int \frac{\sin^3 x}{\cos^4 x}\, dx$

(f/4) $\int \frac{1}{\sinh x \cosh x}\, dx$

(f/5) $\int \cos 3x dx$

(f/6) $\int x \sin x^2 dx$

(f/7) $\int (\sin^2 x - \cos^2 x) dx$

(f/8) $\int \sin x \cos x dx$

(f/9) $\int \sin^2 x \cos^2 x dx$

(f/10) $\int \sin^3 x \cos x dx$

(f/11) $\int \sin^2 x \cos^3 x dx$

(f/12) $\int \sin^5 x dx$

(f/13) $\int \cos^5 x dx$

(f/14) $\int \cos^3 x dx$

(f/15) $\int \sin^5 x \cos^2 x dx$

(f/16) $\int \frac{\sin x dx}{1+\cos^2 x}$

(f/17) $\int \frac{\cos x dx}{1+\cos 2x}$

(f/18) $\int \frac{\sin^3 x dx}{1+\sin^2 x}$

(f/19) $\int \sqrt{1+\cos x} dx$

(f/20) $\int \sqrt{1-\cos x} dx$

(f/21) $\int \frac{dx}{\sin x + \cos x}$

(f/22) $\int \frac{dx}{\cos^3 x}$

(f/23) $\int \frac{dx}{5 - 3\cos x}$

(f/24) $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

(f/25) $\int \mathrm{tg}^5 x dx$

(f/26) $\int \frac{dx}{\sin^4 x \cos^4 x}$

Az $R(e^x)$ alakú függvények integrálása

Számítsuk ki a következő integrálokat:

(g/1) $\int \sinh^2 x dx$

(g/2) $\int \cosh^3 x dx$

(g/3) $\int \frac{dx}{shx}$

(g/4) $\int \sinh^2 xch^3 dx$

(g/5) $\int \frac{\sinh^3 x dx}{\sqrt{\cosh x}}$

(g/6) $\int \frac{e^{2x} dx}{e^x + 1}$

(g/7) $\int \frac{6dx}{e^x - 3}$

(g/8) $\int e^x \sinh 3x dx$

Irracionális függvények integrálása

Számítsuk ki a következő integrálokat:

(h/1) $\int \frac{xdx}{\sqrt{3x+5}}$

(h/2) $\int (x^2 - 3x + 2)\sqrt{2x-1} dx$

(h/3) $\int \frac{dx}{\sqrt{e^x + 1}}$

(h/4) $\int \frac{\sqrt[3]{x^2} dx}{\sqrt{9x^2 - 6x + 5}}$

(h/5) $\int \frac{\sinh x dx}{\sqrt{\cosh x}}$

(h/6) $\int \frac{xdx}{\sqrt{x^2+1}}$

(h/7) $\int \frac{x^3 dx}{\sqrt{x^8 - 1}}$

További feladatok

Számítsuk ki a következő integrálokat:

a) $\displaystyle\int \frac{\cos 2x dx}{\cos x - \sin x}$

b) $\displaystyle\int e^{-x} dx$

c) $\displaystyle\int \cos(4x-5) dx$

d) $\displaystyle\int \sqrt{8-2x} dx$

e) $\displaystyle\int 10^x e^x dx$

f) $\displaystyle\int \frac{5+x^2}{5-x^2} dx$

g) $\displaystyle\int \frac{3dx}{\sqrt{3x^2-2}}$

h) $\displaystyle\int \frac{x^3 dx}{(2x-4)^5}$

i) $\displaystyle\int \sqrt[5]{(8-3x)^6} dx$

j) $\displaystyle\int x\sqrt{1-x^2} dx$

k) $\displaystyle\int \frac{\cos x dx}{\sqrt{\sin x}}$

l) $\displaystyle\int x \sin(x^2+1) dx$

m) $\displaystyle\int \frac{dx}{x \ln x}$

n) $\displaystyle\int \frac{\sqrt{\ln x}}{x} dx$

o) $\displaystyle\int \frac{\cos x dx}{\sqrt{1+\sin^2 x}}$