HELYETTESÍTÉSES INTEGRÁLÁS:

$$\int x^3 \cdot \sqrt{x^2+1}\, dx = \boxed{\begin{array}{l} u = x^2+1 \Rightarrow x^2 = u-1 \\ \frac{du}{dx} = 2x \Longrightarrow x\,dx = \frac{1}{2}\,du \end{array}}$$

$$= \int \underbrace{x^2}_{u-1} \cdot \underbrace{\sqrt{x^2+1}}_{\sqrt{u}} \cdot \underbrace{x\,dx}_{\frac{1}{2}du} = \frac{1}{2}\int (u-1)\cdot\sqrt{u}\,du =$$

$$= \frac{1}{2}\left(\int u^{3/2}\,du - \int u^{1/2}\,du\right) = \frac{1}{2}\left(\frac{2}{5}\cdot u^{5/2} - \frac{2}{3}\cdot u^{3/2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{5}\cdot(x^2+1)^{5/2} - \frac{2}{3}\cdot(x^2+1)^{3/2}\right) + C$$

$$\int \frac{1}{1+\sqrt{x+1}}\,dx = \boxed{\begin{array}{l} u = \sqrt{x+1} \\ \frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x+1}} \end{array} \quad \begin{array}{l} dx = 2\sqrt{x+1}\,du \\ dx = 2u\,du \end{array}}$$

$$= \int \frac{1}{1+u}\,2u\,du = 2\int \frac{u}{1+u}\,du = 2\cdot\int \frac{u+1-1}{u+1}\,du =$$

$$= 2\cdot\int\left(1 - \frac{1}{u+1}\right)du = 2\int 1\,du - 2\int \frac{1}{u+1}\,du =$$

$$= 2u - 2\ln|u+1| + C =$$

$$= 2\cdot\sqrt{x+1} - 2\cdot\ln\left|\sqrt{x+1}+1\right| + C$$

139. OLDAL

$$\int e^{\sqrt{x}}\,dx = \boxed{\begin{array}{l} u=\sqrt{x} \\ \frac{du}{dx}=\frac{1}{2}\frac{1}{\sqrt{x}} \end{array} \quad \begin{array}{l} dx = 2\sqrt{x}\,du \\ dx = 2u\,du \end{array}}$$

$$= \int e^u\, 2u\,du = 2\cdot\int \underbrace{e^u}_{f'}\cdot\underbrace{u}_{g}\,du = \text{PARCIÁLIS INTEGRÁLÁS}$$

$$= 2\cdot\left(\underbrace{e^u}_{f}\cdot\underbrace{u}_{g} - \int \underbrace{e^u}_{f}\cdot\underbrace{1}_{g'}\,du\right) = 2\cdot e^u\cdot(u-1)+C$$

$$= 2\cdot e^{\sqrt{x}}\cdot(\sqrt{x}-1)+C$$

TRIGONOMETRIKUS FÜGGVÉNYEK INTEGRÁLÁSA:

$$\int (\sin(x))^n\cdot(\cos(x))^m\,dx = ?$$

AKKOR KÖNNYŰ KISZÁMOLNI, HA $\boxed{n \text{ VAGY } m \text{ PÁRATLAN}}$

PL: $\int \cos^5(x)\cdot\sin^4(x)\,dx =$ $\boxed{\text{cos(x) KITEVŐJE PÁRATLAN}}$

$$= \int \sin^4(x)\cdot(\cos^2(x))^2\cdot\cos(x)\,dx =$$

$$= \int \sin^4(x)\cdot(1-\sin^2(x))^2\cdot\underbrace{\cos(x)\,dx}_{du}$$

$$\boxed{\begin{array}{l|l} u=\sin(x) & du= \\ \frac{du}{dx}=\cos(x) & \cos(x)\,dx \end{array}}$$

$$= \int u^4\cdot(1-u^2)^2\,du = (\star)$$

$\boxed{\text{140. OLDAL}}$

$$\text{☆} = \int u^4 \cdot (1-u^2)^2 \, du = \int u^4 \cdot (1-2u^2+u^4)\, du =$$

$$= \int (u^4 - 2u^6 + u^8)\, du = \frac{u^5}{5} - 2 \cdot \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\sin(x)^5}{5} - \frac{2}{7}\sin(x)^7 + \frac{\sin(x)^9}{9} + C$$

<u>PL</u>: $\displaystyle\int \frac{\sin^3(x)}{\sqrt{\cos(x)}}\, dx = $ $\boxed{\text{sin(x) KITEVŐJE PÁRATLAN!}}$

$$= \int \frac{\sin^2(x)}{\sqrt{\cos(x)}} \sin(x)\, dx = \int \frac{1-\cos^2(x)}{\sqrt{\cos(x)}} \cdot \underbrace{\sin(x)\, dx}_{(-du)}$$

$$= \boxed{\begin{array}{l} u = \cos(x) \\ \frac{du}{dx} = -\sin(x) \implies -du = \sin(x)dx \end{array}} = \int \frac{1-u^2}{\sqrt{u}} (-du) =$$

$$= \int (u^{3/2} - u^{-1/2})\, du = \frac{1}{1+3/2} \cdot u^{3/2+1} - \frac{1}{1-1/2} \cdot u^{-1/2+1} + C =$$

$$= \frac{2}{5} \cdot u^{5/2} - 2 \cdot u^{1/2} + C = \frac{2}{5} \cdot (\cos(x))^{5/2} - 2 \cdot (\cos(x))^{1/2} + C$$

141. OLDAL

$\int (\sin(x))^n \cdot (\cos(x))^m \, dx = ?$ HA MIND n, MIND m PÁROS, AKKOR HASZNÁLJUK A

$$\boxed{\sin^2(x) = \frac{1-\cos(2x)}{2}} \quad \boxed{\cos^2(x) = \frac{1+\cos(2x)}{2}}$$

AZONOSSÁGOKAT (LÁSD KÉPLETGYŰJTEMÉNY)

PL: $\int \sin^2(x)\cdot\cos^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)\cdot\left(\frac{1+\cos(2x)}{2}\right)dx =$

$= \frac{1}{4}\int (1-\cos^2(2x))\,dx = \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\,dx =$

$= \frac{1}{4}x - \frac{1}{4}\cdot\int \frac{1+\cos(4x)}{2}\,dx =$

$= \frac{1}{4}x - \frac{1}{8}\int 1\,dx - \frac{1}{8}\int \cos(4x)\,dx =$

$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin(4x) + C$

NEVEZETES HELYETTESÍTÉSEK (BEMELEGÍTÉS):

$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx =$

$x = \sin(u)$	$\sqrt{1-x^2} = \sqrt{1-\sin(u)^2} = \cos(u)$
$\frac{dx}{du} = \cos(u)$	$\Rightarrow dx = \cos(u)\,du$

$= \int \frac{\arcsin(\sin(u))}{\cos(u)}\cdot\cos(u)\,du = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\arcsin(x)^2 + C$

142. OLDAL

$$\int \frac{1}{\operatorname{arsh}(x)\cdot\sqrt{1+x^2}}\,dx = \boxed{\begin{array}{l|l} x = \operatorname{sh}(u) & \sqrt{1+x^2} = \\ \frac{dx}{du} = \operatorname{ch}(u) & \sqrt{1+\operatorname{sh}^2(u)} = \\ dx = \operatorname{ch}(u)\,du & \sqrt{\operatorname{ch}^2(u)} = \operatorname{ch}(u) \end{array}}$$

$$= \int \frac{1}{\operatorname{arsh}(\operatorname{sh}(u))\cdot\operatorname{ch}(u)}\cdot\operatorname{ch}(u)\,du$$

$$= \int \frac{1}{u}\,du = \ln(u) + C = \ln\left(\operatorname{arsh}(x)\right) + C$$

NEVEZETES HELYETTESÍTÉSEK:

EZT LÁTOD:	EZT HELYETTESÍTSD:	EZT HASZNÁLD:
$\sqrt{a^2-x^2}$	$x = a\cdot\sin(u)$	$1-\sin^2(x) = \cos^2(x)$
$\sqrt{a^2+x^2}$	$x = a\cdot\operatorname{sh}(u)$	$\operatorname{sh}^2(x)+1 = \operatorname{ch}^2(x)$
$\sqrt{x^2-a^2}$	$x = a\cdot\operatorname{ch}(u)$	$\operatorname{ch}^2(x)-1 = \operatorname{sh}^2(x)$

PL: $\int \sqrt{x^2-2}\,dx = \boxed{\begin{array}{l} x = \sqrt{2}\cdot\operatorname{ch}(u) \\ dx = \sqrt{2}\operatorname{sh}(u)\,du \end{array}}$ $\boxed{\begin{array}{l} \sqrt{x^2-2} = \\ \sqrt{2\operatorname{ch}^2(u)-2} \\ = \sqrt{2}\cdot\sqrt{\operatorname{ch}^2(u)-1} \\ = \sqrt{2}\cdot\operatorname{sh}(u) \end{array}}$

$$= \int \sqrt{2}\operatorname{sh}(u)\cdot\sqrt{2}\operatorname{sh}(u)\,du =$$

$$= 2\int \operatorname{sh}^2(u)\,du \underset{\substack{\uparrow \\ \text{KÉPLETGYŰJT.}}}{=} 2\cdot\int \frac{\operatorname{ch}(2u)-1}{2}\,du = ✪$$

143. OLDAL

$$★ = \int ch(2u)\,du - \int 1\,du = \frac{1}{2} sh(2u) - u + C =$$

$$\underset{\substack{\uparrow \\ \boxed{\text{KÉPLETGYŰJT}}}}{=\!=\!=} \frac{1}{2}\cdot 2\cdot sh(u)\cdot ch(u) - u + C = ☹$$

$$\boxed{x = \sqrt{2}\cdot ch(u) \Rightarrow ch(u) = \frac{x}{\sqrt{2}}}$$

$$\boxed{\sqrt{x^2-2} = \sqrt{2}\cdot sh(u) \Rightarrow sh(u) = \frac{\sqrt{x^2-2}}{\sqrt{2}}}$$

$$\boxed{x = \sqrt{2}\cdot ch(u) \Rightarrow u = arch\left(\frac{x}{\sqrt{2}}\right)}$$

$$☹ = \frac{\sqrt{x^2-2}}{\sqrt{2}}\cdot\frac{x}{\sqrt{2}} - arch\left(\frac{x}{\sqrt{2}}\right) + C$$

$$= \frac{\sqrt{x^2-2}\cdot x}{2} - arch\left(\frac{x}{\sqrt{2}}\right) + C$$

PL.: $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx =$ $\boxed{\begin{array}{l} x = \sin(u) \Rightarrow \sqrt{1-x^2} = \cos(u) \\ \frac{dx}{du} = \cos(u) \Rightarrow dx = \cos(u)\,du \end{array}}$

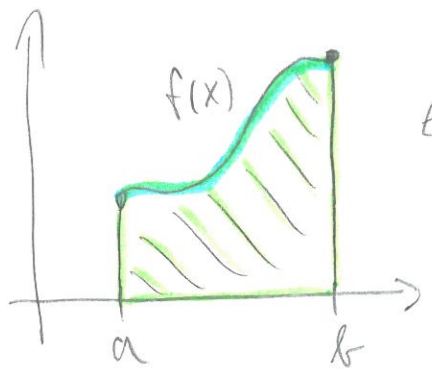
$$= \int \frac{\sin^2(u)}{\cos(u)}(\cos(u)\,du) = \int \sin^2(u)\,du = \boxed{\text{KÉPLET GYŰJT}}$$

$$= \int \frac{1-\cos(2u)}{2}\,du = \frac{u}{2} - \frac{\sin(2u)}{4} + C = \boxed{\text{KÉPLET GYŰJT}}$$

$$= \frac{u}{2} - \frac{2\cdot\sin(u)\cdot\cos(u)}{4} + C = \frac{\arcsin(x)}{2} - \frac{x\cdot\sqrt{1-x^2}}{2} + C$$

144. OLDAL

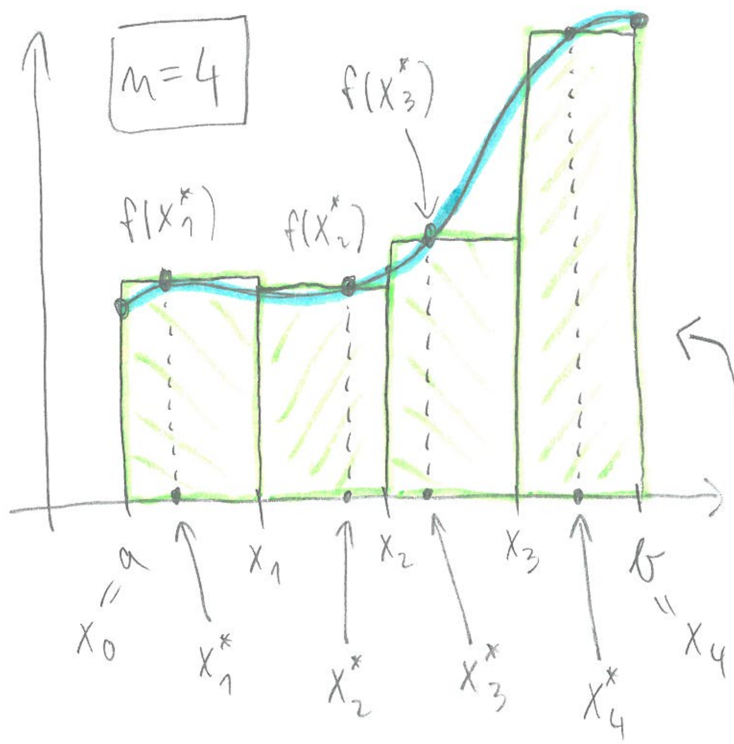
HATÁROZOTT INTEGRÁL:



MI AZ f(x) GÖRBÉJE ALATTI TERÜLET a-TÓL b-IG?

JELÖLÉS: $\boxed{T = \int_a^b f(x)\,dx}$

HOGY SZÁMOLJUK KI? KÖZELÍTSÜK TÉGLALAPOKKAL!



k-ADIK TÉGLALAP TERÜLETE:

ALAP × MAGASSÁG =

$(x_k - x_{k-1}) \cdot f(x_k^*) = T_k$

$R_4 = T_1 + T_2 + T_3 + T_4$

$x_{k-1} \leq x_k^* \leq x_k$

HA n EGYENLŐ RÉSZRE OSZTJUK $[a, b]$-T:

$[\underset{a}{\overset{\shortparallel}{x_0}}, x_1], [x_1, x_2], \dots, [x_{n-1}, \underset{b}{\overset{\shortparallel}{x_n}}]$ HOSSZUK: $\boxed{\Delta x = \frac{b-a}{n}}$

RIEMANN-FÉLE KÖZELÍTŐ ÖSSZEG: $\boxed{x_k = a + k \cdot \Delta x}$

$R_n = \Delta x \cdot f(x_1^*) + \dots + \Delta x \cdot f(x_n^*)$ $\boxed{x_k - x_{k-1} = \Delta x}$

DEFINÍCIÓ: $\int_a^b f(x)\,dx = \lim_{n \to \infty} R_n$

145. OLDAL

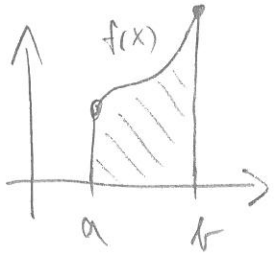
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n$$

AHOGY $n\to\infty$, $\Delta x = \frac{b-a}{n} \to 0$, A FELOSZTÁS EGYRE FINOMODIK, ÉS ÍGY R_n EGYRE PONTOSABBAN KÖZELÍTI A KERESETT TERÜLETET.

$\int_a^b f(x)\,dx$ = f HATÁROZOTT INTEGRÁLJA a-TÓL b-IG

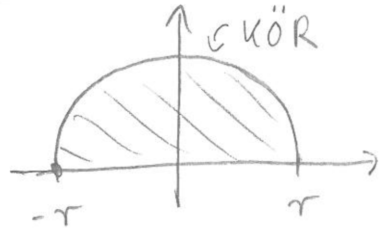
MEGJ: $\int_a^b f(x)\,dx$ NEM x FÜGGVÉNYE, MINT A HATÁROZATLAN INTEGRÁL $\int f(x)\,dx$!

$\int_a^b f(x)\,dx =$



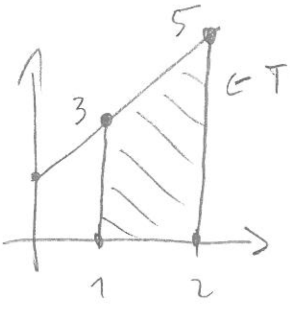
$= \int_a^b f(t)\,dt = \int_a^b f(u)\,du$ STB.

PL: $\int_{-r}^{r} \sqrt{r^2-x^2}\,dx =$



$= \frac{r^2\cdot\pi}{2}$

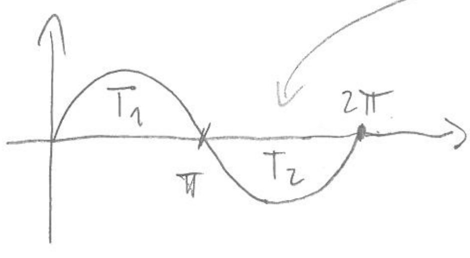
$\int_1^2 (2x+1)\,dx =$



$= (2-1)\cdot\left(\frac{3+5}{2}\right)$

ELŐJELES TERÜLET

$\int_0^{2\pi} \sin(x)\,dx =$



$= T_1 - T_2 = 0$

146. OLDAL