Midterm Exam - May 15, 2024, Limit thms. of probab., SOLUTIONS

1. Let $Y_1, Y_2, \ldots$ denote i.i.d. random variables with distribution $\mathbb{P}(Y_i = +1) = \frac{1}{3}$, $\mathbb{P}(Y_i = -1) = \frac{1}{3}$, $\mathbb{P}(Y_i = 0) = \frac{1}{3}$. Let $Z_0 = 0$ and $Z_n = Y_1 + \cdots + Y_n$. Let $\tau := \min\{n \geq 0 \,:\, Z_n = 1\}$.

   Let $\mathcal{T}_0 := 0$ and let $\mathcal{T}_k := \min\{n > \mathcal{T}_{k-1} \,:\, Z_n = 0\}$.

   (a) Let $z \in \mathbb{C}$ with $|z| \leq 1$. Find $\mathbb{E}[z^\tau]$. *Hint:* You will have to solve a quadratic equation.

   (b) Let $z \in \mathbb{C}$ with $|z| \leq 1$. Find $\mathbb{E}[z^{\mathcal{T}_1}]$.

   (c) Find $\mathbb{E}[z^{\mathcal{T}_k}]$.

   (d) Find the value of $\eta \in \mathbb{R}_+$ such that $\mathcal{T}_k/k^\eta \Rightarrow \mathcal{T}$ as $k \to \infty$ (where $\mathcal{T}$ is a non-degenerate random variable) and find the characteristic function of $\mathcal{T}$.

   Solution:

   (a) Let $G(z) = \mathbb{E}[z^\tau]$. Similarly to page 99-100 of the scanned lecture notes, we have $G(z) = \frac{1}{3}z + \frac{1}{3}zG(z) + \frac{1}{3}zG^2(z)$. Thus $zG^2(z) + (z-3)G(z) + z = 0$, thus $G(z) = \frac{(3-z)-\sqrt{(z-3)^2-4z^2}}{2z}$ (the other solution would give $G(0) = \infty$)

   (b) $\mathbb{E}[z^{\mathcal{T}_1}] = \frac{1}{3}z + \frac{2}{3}zG(z) = 1 - \sqrt{1 - \frac{2}{3}z - \frac{1}{3}z^2}$.

   (c) $\mathbb{E}[z^{\mathcal{T}_k}] = \left(1 - \sqrt{1 - \frac{2}{3}z - \frac{1}{3}z^2}\right)^k$, cf. page 62 or 65 of scanned or HW7.2 for a similar argument.

   (d) Let $Z_k := \mathcal{T}_k/k^\eta$. The characteristic function of Z_k is $\left(1 - \sqrt{1 - \frac{2}{3}e^{it/k^\eta} - \frac{1}{3}e^{i2t/k^\eta}}\right)^k$. We know that if $k \cdot a_k \to c$ then $(1-a_k)^k \to e^{-c}$, so we need to find η so that for any $t \in \mathbb{R}$ the sequence $k \cdot \sqrt{1 - \frac{2}{3}e^{it/k^\eta} - \frac{1}{3}e^{i2t/k^\eta}}$ converges to a finite non-zero value as $k \to \infty$. Therefore we want $\lim_{k\to\infty} k^2 \cdot (1 - \frac{2}{3}e^{it/k^\eta} - \frac{1}{3}e^{i2t/k^\eta}) = b$ to be non-zero and finite. Thus we must choose $\eta = 2$ and we may use L'Hospital's rule to conclude that the limit is $b = -\frac{2}{3}it - \frac{1}{3}2it = -\frac{4}{3}it$. Thus $Z_k \Rightarrow Z$ as $k \to \infty$ where $\mathbb{E}[e^{itZ}] = e^{-\sqrt{-\frac{4}{3}it}}$ (thus Z is a constant times standard Lévy, cf. HW7.2).

2. Let $X_1, X_2, \ldots$ denote independent random variables with the following distribution: $\mathbb{P}(X_k = \pm k^2) = \frac{1}{4\sqrt{k}}$, $\mathbb{P}(X_k = \pm k^3) = \frac{1}{4k^2}$, $\mathbb{P}(X_k = 0) = 1 - \frac{1}{2\sqrt{k}} - \frac{1}{2k^2}$. Let $S_n = X_1 + \cdots + X_n$.

   (a) Show that Lindeberg's theorem cannot be applied to the above case in order to prove $\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \Rightarrow \mathcal{N}(0,1)$ because Lindeberg's condition fails.

   (b) Find a, b, α, β such that $\frac{S_n - an^\alpha}{bn^\beta} \Rightarrow \mathcal{N}(0,1)$. *Hint:* use truncation, Borel-Cantelli and Lindeberg (for the truncated random variables).

   Hint: In your calculations you may use without proof that for any $\gamma > -1$ we have $1^\gamma + 2^\gamma + \cdots + n^\gamma \approx \frac{n^{\gamma+1}}{\gamma+1}$.

   Solution:

   (a) $\mathbb{E}[X_k] = 0$, $\mathrm{Var}(X_k) = \frac{1}{2}k^{3.5} + \frac{1}{2}k^4 \leq k^4$. $\mathrm{Var}(S_n) \leq \sum_{k=1}^n k^4 \leq n^5$. Thus $\sigma_n = \sqrt{\mathrm{Var}(S_n)} \leq n^{2.5}$. We will show that Lindeberg's condition fails for $\varepsilon = 1$. Note that if n is large enough then $\left(\frac{n}{2}\right)^3 > \varepsilon\sigma_n$, thus $\frac{1}{\sigma_n^2}\sum_{k=1}^n \mathbb{E}\big[|X_k|^2 \cdot \mathbb{1}\,[|X_k| > \varepsilon\sigma_n]\big] \geq n^{-5}\sum_{k=n/2}^n k^6 \frac{1}{2}\frac{1}{k^2} \geq n^{-5}\frac{n}{2}\left(\frac{n}{2}\right)^6 \frac{1}{2}\frac{1}{n^2} = 2^{-8}$. The r.h.s. does not go to zero as $n \to \infty$, thus Lindeberg's condition fails.

   (b) Let $\widetilde{X}_k = X_k \cdot \mathbb{1}[\,|X_k| \leq k^2\,]$ and $\widetilde{S}_n = \widetilde{X}_1 + \cdots + \widetilde{X}_n$. $\mathbb{E}[\widetilde{X}_k] = 0$, $\mathrm{Var}(\widetilde{X}_k) = \frac{1}{2}k^{3.5}$, $\mathrm{Var}(\widetilde{S}_n) = \sum_{k=1}^n \frac{1}{2}k^{3.5} \approx \frac{n^{4.5}}{9}$, $\widetilde{\sigma}_n = \sqrt{\mathrm{Var}(\widetilde{S}_n)} \approx \frac{n^{2.25}}{3}$, thus $a = 0$, α can be anything, $b = \frac{1}{3}$ and $\beta = 2.25$. Now we check Lindeberg's condition. For any $k \leq n$ we have $|\widetilde{X}_k| \leq k^2 \leq n^2$, thus for any $\varepsilon > 0$ we have $\mathbb{1}\left[|\widetilde{X}_k| > \varepsilon\widetilde{\sigma}_n\right] = 0$ if n is large enough. Thus $\widetilde{\sigma}_n^{-2}\sum_{k=1}^n \mathbb{E}\left[|\widetilde{X}_k|^2 \cdot \mathbb{1}\left[|\widetilde{X}_k| > \varepsilon\widetilde{\sigma}_n\right]\right] \to 0$ as $n \to \infty$. Lindeberg states $\frac{\widetilde{S}_n - \mathbb{E}(\widetilde{S}_n)}{\sqrt{\mathrm{Var}(\widetilde{S}_n)}} \Rightarrow \mathcal{N}(0,1)$, thus (multiplicative) Slutsky implies $\frac{\widetilde{S}_n}{bn^\beta} \Rightarrow \mathcal{N}(0,1)$.

   Now let us observe that $\mathbb{P}(\widetilde{X}_k \neq X_k) = \frac{1}{2}\frac{1}{k^2}$, this is summable in k, thus Borel-Cantelli implies that there are only finitely many values of k for which $\widetilde{X}_k \neq X_k$. This implies that $S_n - \widetilde{S}_n$ converges to an almost surely finite random variable as $n \to \infty$, thus $\frac{S_n - \widetilde{S}_n}{bn^b} \to 0$, thus (additive) Slutsky implies the desired

$$\frac{S_n}{bn^\beta} \Rightarrow \mathcal{N}(0,1).$$