Make-up Midterm Exam 1 - April 13, 2022, Limit thms. of probab.

1. Let $Z_1, Z_2, \dots$ denote i.i.d. random variables with p.d.f. $f(x) = 4e^{4x}\mathbb{1}[x<0]$. Let $Y_n := \frac{Z_1+\cdots+Z_n}{n}$.

   (a) (4 points) Let $g_n(x)$ denote the p.d.f. of Y_n. Find all of the possible values of $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $c \in (0, +\infty)$ such that
$$\lim_{n\to\infty} g_n(x) \cdot n^\alpha = c. \tag{1}$$

   (b) (3 points) Calculate $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left[Y_n \le x\right]\right)$ for each $x \in \mathbb{R}$.

   Solution: Note that if $Z_i^* := -4Z_i$ then $Z_i^* \sim \text{EXP}(1)$. If we let $S_n^* = Z_1^* + \cdots + Z_n^*$ then $Y_n = -\frac{1}{4}S_n^*/n$. Let $g_n^*(x)$ denote the p.d.f. of S_n^*/n. Thus we have $g_n(x) = 4g_n^*(-4x)$. Note that $\mathbb{E}(Y_n) = -\frac{1}{4}$.

   (a) We know from the solution of HW3.2(e) that $g_n^*(x) \approxeq e^{-nI(x)}$, where $I(x) = x - 1 - \ln(x)$ is the large deviation rate function of the EXP(1) distribution. We know that $I(x) > 0$ if $x \neq 1$, thus $g_n^*(x)$ decays exponentially as $n \to \infty$ for any $x \neq 1$. By $g_n(x) = 4g_n^*(-4x)$ we obtain that $g_n(x)$ goes to zero exponentially fast if $x \neq -\frac{1}{4}$. Thus if $x \neq -\frac{1}{4}$ then $\lim_{n\to\infty} g_n(x) \cdot n^\alpha = 0$ for any $\alpha \in \mathbb{R}$. Thus our only chance for achieving (1) with a positive c is if we choose $x = -\frac{1}{4}$. Let $X_n := \frac{S_n^* - n}{\sqrt{n}}$ Let $f_n(x)$ denote the p.d.f. of X_n. We know from class (see page 47-50) that $f_n(x) \to \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ as $n \to \infty$, thus $f_n(0) \approx \frac{1}{\sqrt{2\pi}}$. Note that $S_n^*/n = \frac{X_n}{\sqrt{n}} + 1$, thus $g_n^*(x) = \sqrt{n} f_n(\sqrt{n}(x-1))$, in particular $g_n^*(1) \approx \sqrt{n}\frac{1}{\sqrt{2\pi}}$. Thus $g_n(-\frac{1}{4}) = 4g_n^*(1) \approx \sqrt{n}\frac{4}{\sqrt{2\pi}}$. Thus $x = -\frac{1}{4}$, $\alpha = -\frac{1}{2}$ and $c = \frac{4}{\sqrt{2\pi}}$ in (1).

   (b)
$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left[Y_n \le x\right]\right) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left[S_n^*/n \ge -4x\right]\right) \overset{(*)}{=} -\min_{y \ge -4x} I(y) = \begin{cases} -I(1) = 0 & \text{if } x \ge -\frac{1}{4}, \\ -I(-4x) & \text{if } x < -\frac{1}{4}, \end{cases}$$
   where $(*)$ holds by Cramér's theorem with $I(\cdot)$ defined above.

2. Let $p \in (0, \frac{1}{2})$. Let $X_1, X_2, \dots$ denote i.i.d. random variables with distribution
$$\mathbb{P}(X_i = +1) = p, \qquad \mathbb{P}(X_i = -1) = p, \qquad \mathbb{P}(X_i = 0) = 1 - 2p. \tag{2}$$
   Let $W_0 = 0$ and $W_n = X_1 + \cdots + X_n$. Let $V_n = \max\{W_0, W_1, \dots, W_n\}$. Let $\tau_\ell := \min\{n : W_n = \ell\}$.

   (a) (3 points) Find $\beta \in \mathbb{R}_+$ such that W_n/n^β converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   (b) (3 points) Find $\gamma \in \mathbb{R}_+$ such that V_n/n^γ converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   (c) (2 points) Find $\delta \in \mathbb{R}_+$ such that τ_ℓ/ℓ^δ converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   Solution:

   (a) $\mathbb{E}(X_i) = 0$, $\text{Var}(X_i) = \mathbb{E}(X_i^2) = 2p$. Thus by the CLT we have $W_n/\sqrt{2pn} \Rightarrow \mathcal{N}(0,1)$, therefore $\beta = 1/2$ and $W_n/n^\beta \Rightarrow \mathcal{N}(0, \sqrt{2p}^2)$, thus the c.d.f. of the limiting distribution of W_n/n^β is $\Phi(\frac{x}{\sqrt{2p}})$.

   (b) For any $k \in \mathbb{N}_0$ we have $\mathbb{P}(V_n \ge k) = 2\mathbb{P}(W_n > k) + \mathbb{P}(W_n = k)$ by the reflection principle (i.e., by the same argument as on page 58, which only used that the increments of the walk are i.i.d., symmetric and have absolute value less than or equal to 1, and all of these properties hold for (2)). Thus if we pick $\gamma = 1/2$ then for any $x \ge 0$ we have
$$\mathbb{P}(V_n/n^\gamma \ge x) = \mathbb{P}(V_n \ge \lceil xn^\gamma \rceil) \to 2 \cdot (1 - \Phi(\frac{x}{\sqrt{2p}})),$$
   since $2\mathbb{P}(W_n > \lceil xn^\gamma \rceil + 1) \le \mathbb{P}(V_n \ge \lceil xn^\gamma \rceil)) \le 2\mathbb{P}(W_n > \lceil xn^\gamma \rceil)$, and we have $\mathbb{P}(W_n > \lceil xn^\gamma \rceil) \to 1 - \Phi(\frac{x}{\sqrt{2p}})$ and $\mathbb{P}(W_n > \lceil xn^\gamma \rceil + 1) \to 1 - \Phi(\frac{x}{\sqrt{2p}})$ by part (a) and Slutsky. Thus $V_n/\sqrt{n}$ weakly converges and the c.d.f. of the limit distribution is $2\Phi(\frac{x}{\sqrt{2p}}) - 1$ for $x \ge 0$ (and zero otherwise).

   (c) $\{\tau_\ell \le n\} = \{V_n \ge \ell\}$, thus for any $x > 0$ we have $\delta = 2$ and $\mathbb{P}(\tau_\ell/\ell^2 \le x) = \mathbb{P}(\tau_\ell \le \lfloor \ell^2 x \rfloor) = \mathbb{P}(V_{\lfloor \ell^2 x \rfloor} \ge \ell) = \mathbb{P}(V_{\lfloor \ell^2 x \rfloor}/\sqrt{\lfloor \ell^2 x \rfloor} \ge \ell/\sqrt{\lfloor \ell^2 x \rfloor}) \to 2 \cdot (1 - \Phi(\frac{1}{\sqrt{2px}}))$ by part (b) and Slutsky. Thus the c.d.f. of the weak limit of τ_ℓ/ℓ^δ is $2 \cdot (1 - \Phi(\frac{1}{\sqrt{2px}}))$ if $x > 0$ and zero otherwise.