Make-up Midterm Exam 1 - April 13, 2022, Limit thms. of probab.

Family name ——————————— **Given name** ————————————

Signature ——————————— **Neptun Code** ————————————

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $Z_1, Z_2, \ldots$ denote i.i.d. random variables with p.d.f. $f(x) = 4e^{4x}\mathbb{1}[x < 0]$. Let $Y_n := \frac{Z_1 + \cdots + Z_n}{n}$.

   (a) (4 points) Let $g_n(x)$ denote the p.d.f. of Y_n. Find all of the possible values of $x \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $c \in (0, +\infty)$ such that
   $$\lim_{n \to \infty} g_n(x) \cdot n^{\alpha} = c.$$

   (b) (3 points) Calculate $\lim_{n \to \infty} \frac{1}{n} \ln \left(\mathbb{P}\left[Y_n \leq x \right] \right)$ for each $x \in \mathbb{R}$.

   Instruction: For both (a) and (b), you do not have to carry out lengthy calculations, it is enough to refer to results learnt in class or results of homework exercises.

2. Let $p \in (0, \frac{1}{2})$. Let $X_1, X_2, \ldots$ denote i.i.d. random variables with distribution
   $$\mathbb{P}(X_i = +1) = p, \qquad \mathbb{P}(X_i = -1) = p, \qquad \mathbb{P}(X_i = 0) = 1 - 2p.$$

   Let $W_0 = 0$ and $W_n = X_1 + \cdots + X_n$. Let $V_n = \max\{W_0, W_1, \ldots, W_n\}$. Let $\tau_\ell := \min\{n : W_n = \ell\}$.

   (a) (3 points) Find $\beta \in \mathbb{R}_+$ such that W_n/n^{β} converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   (b) (3 points) Find $\gamma \in \mathbb{R}_+$ such that V_n/n^{β} converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   (c) (2 points) Find $\delta \in \mathbb{R}_+$ such that τ_ℓ/ℓ^{δ} converges in distribution to a non-degenerate random variable. Write down the c.d.f. of the limiting distribution.

   Instruction: For (a), (b) and (c), please express your answer in terms of $\Phi(\cdot)$, the c.d.f. of the standard normal distribution. Please avoid lengthy calculations, it is enough to refer to results stated in class or results of homework exercises.