Midterm Exam - March 27, 2024, Limit thms. of probab.

Family name ______________________________ **Given name** ______________________________

Signature ______________________________ **Neptun Code** ______________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $X_1, X_2, \dots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(X_i = k) = \frac{2}{3^k}, \qquad k = 1, 2, 3, \dots$$

Let us define $S_n = X_1 + \cdots + X_n$.

(a) Show that

$$\mathbb{P}(S_n = k) = \binom{k-1}{n-1} \frac{2^n}{3^k}, \qquad k = n, n+1, n+2, \dots$$

(b) Calculate

$$\lim_{n \to \infty} \frac{1}{n} \ln \left(\mathbb{P}(S_n = \lfloor nx \rfloor) \right), \qquad x \in \mathbb{R}.$$

(c) Briefly explain how this relates to Cramér's theorem and one of the formulas from the *Formula sheet: large deviation rate functions, exponential tilting*

2. Let $Z_1, Z_2, \dots$ denote i.i.d. random variables with p.d.f.

$$f(x) = x e^{-x} \mathbb{1}[x \geq 0].$$

Let

$$M_n := \max\{Z_1, \dots, Z_n\}.$$

Let us define

$$c_n := \ln(n) + \ln(\ln(n)).$$

Let

$$Y_n := M_n - c_n.$$

Show that Y_n weakly converges as $n \to \infty$ and identify the limiting distribution.