Limit Large Dev. Thms. Exam, June 8, 2022.

1. Let $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$, $0 < p < 1$. Let $0 \leq x \leq 1$. Calculate
$$I(x) = \lim_{n\to\infty} \frac{-1}{n} \ln\left(\mathbb{P}\left[\, S_n = \lfloor n \cdot x \rfloor \,\right]\right)$$
using the crude Stirling formula $k! \approxeq k^k e^{-k}$. Sketch the graph of the function $x \mapsto I(x)$.

   Solution: See page 3 of scanned for the sketch of the graph if I and page 5 for the calculation.

2. Let $F(x) = \mathbb{P}(X \leq x)$. Note that F is right-continuous. Let
$$G(x) = \frac{1}{2}\lim_{\varepsilon\to 0} F(x-\varepsilon) + \frac{1}{2}\lim_{\varepsilon\to 0} F(x+\varepsilon) = \frac{1}{2}\left(F(x_-) + F(x)\right), \qquad x \in \mathbb{R}$$
In particular, if x is a point of continuity of F then $G(x) = F(x)$. Let $\psi(t) = \mathbb{E}(e^{itX})$. Let Y denote an independent random variable with standard normal distribution. For $\sigma \in \mathbb{R}_+$ let F_σ denote the cumulative distribution function of X_σ, where $X_\sigma := X + \sigma Y$.

   (a) Show that $F_\sigma(x) = \mathbb{E}(\Phi\left(\frac{x-X}{\sigma}\right))$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,\mathrm{d}y$.

   (b) Give a formula for $\psi_\sigma(t) := \mathbb{E}(e^{itX_\sigma})$ in terms of ψ.

   (c) Use the dominated convergence theorem to show that $\lim_{\sigma\to 0} F_\sigma(x) = G(x)$.

   (d) For any $a \leq b \in \mathbb{R}$ give an integral formula for $F_\sigma(b) - F_\sigma(a)$ in terms of ψ.
   Hint: Use Fubini and the inversion formula $F'_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_\sigma(t)\,\mathrm{d}t$.

   (e) Use the above results to show that for any $a \leq b \in \mathbb{R}$ we have
$$\frac{1}{2}\mathbb{P}(X = a) + \mathbb{P}(a < X < b) + \frac{1}{2}\mathbb{P}(X = b) = \lim_{\sigma\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ibt} - e^{-iat}}{-it} e^{-\sigma^2 t^2/2}\psi(t)\,\mathrm{d}t.$$

   Solution: See the official solution of homework 8.1.

3. Let us denote by Y_i the highest water level of a river in year i in a city. Let us assume that the random variables $Y_1, Y_2, \ldots$ are independent and identically distributed, moreover let us assume that Y_i has exponential distribution with parameter $\lambda = 2$.

   A city near the river has a dam to protect the city from being flooded. If the city gets flooded in year i, they build a higher dam by the end of the year. If the city does not get flooded, they keep the current dam. Denote by X_i the height of the dam at the end of year i. In year zero, the height of the dam is zero, i.e., $X_0 = 0$. If there was a flood in year i (i.e., the highest water level that year was higher than the dam) then they build a new dam by the end of that year whose height is equal to the highest water level that year.

   (a) Write down a formula for the characteristic function of X_i which contains no integrals.

   (b) Denote by Z_i the number of times they had to rebuild the dam by the end of year i. Find positive real numbers a_i and b_i such that $(Z_i - a_i)/b_i$ converges in distribution to a non-trivial limit as $i \to \infty$, and find the limiting distribution.
   Instruction: Find an expression for a_i and b_i which contains no sums. If you use a theorem learnt in class, please check its conditions.

   Solution: (a) $\mathbb{E}(e^{itX_n}) = \prod_{k=1}^{n}(1 - \frac{it}{2k})^{-1}$, see page 96-97 of scanned, (b) RECORDS: see page 119-120.

4. Let $X_1, X_2, \ldots$ i.i.d. with p.d.f.
$$f(x) = \frac{\alpha}{2|x|^{\alpha+1}}\mathbb{1}[\,|x| > 1\,], \qquad 0 < \alpha < 2.$$
Let $S_n = X_1 + \cdots + X_n$ and let $Z_n = S_n/n^{1/\alpha}$. Prove that
$$\lim_{n\to\infty}\mathbb{E}(e^{itZ_n}) = e^{-c|t|^\alpha}, \qquad c = \alpha\int_0^\infty \frac{1 - \cos(y)}{y^{\alpha+1}}\,\mathrm{d}y.$$
Instruction: Indicate the points in the calculation where the condition $\alpha < 2$ is used.

   Solution: See page 148-151 of scanned.