Limit Large Dev. Thms. Exam, June 5, 2018.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. (a) (5 marks) Prove that if $\mathbb{P}(a \le Y \le b) = 1$ then $\mathrm{Var}(Y) \le \frac{(b-a)^2}{4}$.
   (b) (10 marks) Prove that if $\mathbb{P}(a \le X \le b) = 1$ and $\mathbb{E}(X) = 0$ then $\mathbb{E}(e^{\lambda X}) \le \exp\left(\frac{1}{8}\lambda^2(b-a)^2\right)$.
   (c) (10 marks) State and prove Hoeffding's inequality.

2. Let (X_n) denote one dimensional simple symmetric random walk starting at $X_0 = 0$.
   Let $M_n = \max\{ X_0, X_1, \dots, X_n \}$ denote the running maximum of the walk at time n.
   Let $T_1 = \min\{ n : X_n = 1 \}$ denote the first time when the walker reaches level 1.
   (a) (12 marks) Prove that $\mathbb{P}(M_n \ge k) = 2\mathbb{P}(X_n > k) + \mathbb{P}(X_n = k)$ for any $k = 0, 1, 2, \dots$.
   (b) (13 marks) Use (a) to show that $\mathbb{P}(T_1 > n) = \mathbb{P}(X_n = 0) + \mathbb{P}(X_n = 1)$.

3. (a) (8 marks) State Lindeberg's theorem.
   (b) (17 marks) Use (a) to prove that if $X_1, X_2, \dots$ are i.i.d. with p.d.f. $f(x) = |x|^{-3}\mathbb{1}[\,|x| > 1\,]$ then

$$\frac{X_1 + \cdots + X_n}{\sqrt{n \ln(n)}} \Rightarrow \mathcal{N}(0,1).$$

   Hint: Let $\alpha_n = \sqrt{n}\ln(\ln(n))$ and define $\xi_{n,k} = X_k \mathbb{1}[\,|X_k| < \alpha_n\,]$ for $1 \le k \le n$.

4. Let $\varphi(t) = \mathbb{E}(e^{itX})$.
   (a) (4 marks) Define the notion of a stable distribution.
   (b) (4 marks) State the characterization theorem of symmetric stable distributions using characteristic functions. What is the *index* of a stable distribution?
   (c) (5 marks) Prove that if the characteristic function of X is of the special form that appears in the answer to question (b) then the distribution of X is indeed *symmetric* and *stable*.
   (d) (3 marks) Name three famous symmetric stable distributions and identify their index.
   (e) (5 marks) Prove that if $\varphi(t)$ is of the special form that appears in the answer to question (b) and the index α satisfies $\alpha \in (0, 1]$ then $\varphi(t)$ is indeed the characteristic function of a random variable.
   Hint: It is enough to recall (i.e., precisely state) and use the result of Homework 9.3.
   (f) (4 marks) Show that if the index α satisfies $\alpha > 2$ and if $\varphi(t)$ is of the special form that appears in the answer to question (b) then $\varphi(t)$ is not the characteristic function of a random variable.
   Hint: It is enough to write down the short „naive" proof learnt in class.