Limit Large Dev. Thms. Exam, June 1, 2022.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. (a) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := X_1 + X_2$, where X_1 and X_2 are i.i.d. copies of X. Find the Legendre transform of the logarithmic moment generating function of Y.

   (b) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := aX + b$ (where $a, b \in \mathbb{R}$). Find the Legendre transform of the log. mom. gen. function of Y.

   (c) Let $Y_1, Y_2, \dots$ denote i.i.d. integer-valued random variables with distribution

   $$\mathbb{P}(Y_i = -1) = 1/4, \quad \mathbb{P}(Y_i = 0) = 1/2, \quad \mathbb{P}(Y_i = 1) = 1/4.$$

   Find $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left(Y_1 + \cdots + Y_n \geq nx\right)\right)$ for any $x \in \mathbb{R}$.
   Hint: For (c) you can use Cramér's theorem and the explicit formulas for the large deviation rate functions of famous random variables (see handout).

2. (a) Define when the sequence $X_1, X_2, \dots$ of random variables is tight.

   (b) State Helly's theorem about tightness (but don't prove it).

   (c) State Lévy's continuity theorem and show that if the sequence $X_1, X_2, \dots$ satisfies the conditions of the theorem then $X_1, X_2, \dots$ is tight. *Instruction:* State, but do not prove the lemma which relates the tail probabilities of a random variable to the behaviour of its characteristic function near zero.

3. (a) Let $X_1, X_2, \dots$ denote i.i.d. random variables with EXP(1) distribution. Let $M_n := \max\{X_1, \dots, X_n\}$. Show that $M_n - \ln(n)$ weakly converges to a random variable Y.

   (b) Let $Y_n \sim \mathrm{EXP}(n)$. Let $Z_n := Y_n - 1/n$. Let $Z := Z_1 + Z_2 + \dots$
   State and prove an explicit formula about the cumulative distribution function of Z using (a).

   (c) Calculate the characteristic function of Y (which involves the Gamma function) and the characteristic function of Z (which is of product form).

   (d) Put the above results together to obtain $\Gamma(1 - it) = e^{it\gamma} \prod_{n=1}^{\infty} \left(1 - \frac{it}{n}\right)^{-1} \cdot e^{-it/n}$.

4. Let $X_1, X_2, \dots$ i.i.d. with p.d.f.

   $$f(x) = \frac{\alpha}{2|x|^{\alpha+1}} \mathbb{1}[\,|x| > 1\,], \qquad 0 < \alpha < 2.$$

   Let $S_n = X_1 + \cdots + X_n$ and let $Z_n = S_n / n^{1/\alpha}$. Prove that

   $$\lim_{n\to\infty} \mathbb{E}(e^{itZ_n}) = e^{-c|t|^{\alpha}}, \qquad c = \alpha \int_0^{\infty} \frac{1 - \cos(y)}{y^{\alpha+1}} \, \mathrm{d}y.$$