Limit/large dev. thms. exercise sheet

March 6, 2024

1. For each $n = 1, 2, \dots$, let $(X_1^n, X_2^n, \dots, X_n^n)$ denote a random vector which is uniformly distributed on the surface of the n-dimensional Euclidean ball of radius $\sqrt{n}$ about the origin. Show that
$$X_1^n \Rightarrow \mathcal{N}(0,1), \qquad n \to \infty$$

2. $\lim_{n\to\infty} e^{-n} \left(\frac{n^0}{0!} + \frac{n^1}{1!} + \dots + \frac{n^n}{n!} \right) =?$

   Note: $e^{-n} \left(\frac{n^0}{0!} + \frac{n^1}{1!} + \frac{n^2}{2!} + \dots \right) = 1.$

3. Let $S_n \sim \mathrm{BIN}(n, \frac{1}{2})$, let $p_n(k) := \mathbb{P}(S_n = k)$, $k = 0, 1, \dots, n$.

   In HW 4.3 you will show that for any $x \in \mathbb{R}$ we have
$$\lim_{n\to\infty} \frac{\sqrt{n}}{2} p_n \left(\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2} x \rfloor \right) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

   Show that this implies:
$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - n/2}{\sqrt{n}/2} \Rightarrow \mathcal{N}(0,1).$$