Limit/large dev. thms. HW assignment 6, SOLUTION

1. Let $f : [0,1] \to \mathbb{R}$ be continuous. Find the value of the following limits

$$\text{a)} \lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_1\, dx_2\, \cdots\, dx_n$$
$$\text{b)} \lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2\, \cdots\, dx_n$$

Hint: There is a probabilistic interpretation to this exercise. Use the equivalent characterization of weak convergence given on page 81 of the scanned lecture notes, similarly to the exercise solved on page 84 of the scanned lecture notes.

Solution: Note that $[0,1]$ is a compact set, so the continuous function f is also bounded.

Let $X_1, \ldots, X_n$ denote i.i.d. UNI$[0,1]$ random variables.

(a) $\int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) dx_1\, dx_2\, \cdots\, dx_n = \mathbb{E}\left(f\left(\frac{X_1+\cdots+X_n}{n}\right)\right)$.
Note that by the weak law of large numbers we have $\frac{X_1+\cdots+X_n}{n} \Rightarrow \frac{1}{2}$ as $n \to \infty$. Note that $0 \le \frac{X_1+\cdots+X_n}{n} \le 1$, thus

$$\lim_{n\to\infty} \mathbb{E}\left(f\left(\frac{X_1+\cdots+X_n}{n}\right)\right) \overset{(*)}{=} \mathbb{E}\left(f\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{2}\right),$$

where in $(*)$ we applied the theorem stated on page 81 of the scanned lecture notes, similarly to the example solved on page 84.

(b) $\int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) dx_1\, dx_2\, \cdots\, dx_n = \mathbb{E}\left[f\left((X_1 X_2 \cdots X_n)^{1/n}\right)\right]$.
Taking the logarithm of the geometric mean we obtain the arithmetic mean of the logarithms, then we apply the weak law of large numbers:

$$\ln\left((X_1 X_2 \cdots X_n)^{1/n}\right) = \frac{\ln(X_1) + \cdots + \ln(X_n)}{n} \Rightarrow \mathbb{E}(\ln(X_k)) = \int_0^1 \ln(x)\, \mathrm{d}x = -1$$

Therefore $(X_1 X_2 \cdots X_n)^{1/n} \Rightarrow e^{-1}$ as $n \to \infty$. Note that $0 \le (X_1 X_2 \cdots X_n)^{1/n} \le 1$, thus

$$\lim_{n\to\infty} \mathbb{E}\left[f\left((X_1 X_2 \cdots X_n)^{1/n}\right)\right] \overset{(*)}{=} \mathbb{E}\left[f\left(e^{-1}\right)\right] = f\left(e^{-1}\right),$$

where in $(*)$ we applied the theorem stated on page 81 of the scanned lecture notes, similarly to the example solved on page 84.

2. *Characteristic function of (a) symmetrized exponential density function and (b) „rooftop" density function.*

Compute the characteristic function of the absolute continuous probability distributions with the following probability density functions on $\mathbb{R}$:

$$\text{(a)} \quad \frac{a}{2} \exp^{-a|x|}, \qquad \text{(b)} \quad \max\{\, a(1-a\,|x|),\, 0\,\},$$

where a is a positive constant.

Hint: This exercise can be viewed a painful integration exercise, but if you are clever enough, you can solve it without calculating integrals (but you will have to use results and tricks from class, see page 85-89 of the scanned lecture notes)!

Solution:

(a) $a \exp^{-ax} \mathbb{1}[x \geq 0]$ is the p.d.f. of the EXP(a) distribution, so if $Y \sim$ EXP(a) and X is independent from Y with

$$\mathbb{P}(X = +1) = \mathbb{P}(X = -1) = \frac{1}{2}$$

then the probability density function of XY has density function $\frac{a}{2} \exp^{-a|x|}$:

$$\mathbb{P}\left(XY \in [x, x + \mathrm{d}x]\right) = \frac{1}{2}\mathbb{P}\left(Y \in [x, x + \mathrm{d}x]\right) + \frac{1}{2}\mathbb{P}\left(Y \in [-x - \mathrm{d}x, -x]\right) =$$
$$\frac{1}{2}\mathbb{P}\left(Y \in [|x|, |x| + \mathrm{d}x]\right) = \frac{a}{2} \exp^{-a|x|} \,\mathrm{d}x$$

Recall that the characteristic function of Y is $\varphi_Y(t) = \frac{1}{1-it/a}$, see page 86 of the scanned lecture notes. So we need to calculate the characteristic function of XY, which is

$$\varphi_{XY}(t) = \mathbb{E}[e^{itXY}] = \frac{1}{2}\mathbb{E}[e^{itY}] + \frac{1}{2}\mathbb{E}[e^{-itY}] =$$
$$\frac{1}{2}\varphi_Y(t) + \frac{1}{2}\varphi_Y(-t) = \frac{1}{2}\frac{1}{1 - it/a} + \frac{1}{2}\frac{1}{1 + it/a} = \frac{1}{1 + (t/a)^2}.$$

(b) This density function is shaped like a „rooftop" (the width of the roof is $2/a$, the height is a). One obtains a random variable with this density function by adding two independent UNI$[-\frac{1}{2a}, \frac{1}{2a}]$ random variables.

Indeed, if $Z = X + Y$, where X and Y are i.i.d. with UNI$[-\frac{1}{2a}, \frac{1}{2a}]$ distribution, then the p.d.f. of X and Y are the same: $f_X(x) = f_Y(x) = a \cdot \mathbb{1}[-\frac{1}{2a} \leq x \leq \frac{1}{2a}]$, thus the p.d.f. of Z is

$$f_Z(x) = \int_{-\infty}^{\infty} f_X(y) f_Y(x - y)\,\mathrm{d}y = \int_{-\infty}^{\infty} a^2 \mathbb{1}[-\frac{1}{2a} \leq y \leq \frac{1}{2a},\; -\frac{1}{2a} \leq x - y \leq \frac{1}{2a}]\,\mathrm{d}y =$$
$$\int_{-\frac{1}{2a}\vee(-\frac{1}{2a}+x)}^{\frac{1}{2a}\wedge(\frac{1}{2a}+x)} a^2 \,\mathrm{d}x = \max\{\, a(1 - a\,|x|),\, 0\,\},$$

thus we only need to calculate the characteristic function of Z, where $Z = X + Y$, which is

$$\varphi_Z(t) = \varphi_X(t)\varphi_Y(t) = (\varphi_X(t))^2 \stackrel{(*)}{=} \left(\frac{\sin(t/2a)}{t/2a}\right)^2,$$

where in equation $(*)$ we substituted the characteristic function of UNI$[-\frac{1}{2a}, \frac{1}{2a}]$, calculated in class, see page 89 of lecture notes.

3. *Explicit error bounds for the relation of characteristic functions and moments*

   (a) Show that if $f : \mathbb{R} \to \mathbb{C}$ is three times continuously differentiable then

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \frac{1}{2}\int_0^x f'''(u)(x-u)^2 \,\mathrm{d}u.$$

   (b) Show that $e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} = \frac{i^3}{2}\int_0^x (x-u)^2 e^{iu}\,\mathrm{d}u$

   (c) Show that $e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} = -\int_0^x (x-u)(e^{iu}-1)\,\mathrm{d}u$

   (d) Show that for all $x \in \mathbb{R}$ we have $\left| e^{ix} - \sum_{k=0}^{2} \frac{(ix)^k}{k!} \right| \leq \min\left\{ \frac{|x|^3}{6},\, |x|^2 \right\}$

   (e) Show that if $\varphi(t)$ is the characteristic function of the random variable X and $\mathbb{E}(X^2) < +\infty$ then

$$\left| \varphi(t) - 1 - it\mathbb{E}(X) + \frac{t^2\mathbb{E}(X^2)}{2} \right| \leq \mathbb{E}\left(\min\left\{ \frac{|t|^3}{6}|X|^3,\, t^2|X|^2 \right\}\right)$$

Solution:

(a) $f''(x) = f''(0) + \int_0^x f'''(u)\,\mathrm{d}u$, thus

$$f'(x) = f'(0) + \int_0^x f''(u)\,\mathrm{d}u = f'(0) + \int_0^x \left(f''(0) + \int_0^u f'''(v)\,\mathrm{d}v \right)\mathrm{d}u =$$

$$f'(0) + f''(0)x + \int_0^x \int_0^u f'''(v)\,\mathrm{d}v\,\mathrm{d}u = f'(0) + f''(0)x + \int_0^x (x-v) f'''(v)\,\mathrm{d}v$$

$$f(x) = f(0) + \int_0^x f'(u)\,\mathrm{d}u = f(0) + \int_0^x \left(f'(0) + f''(0)u + \int_0^u (u-v)f'''(v)\,\mathrm{d}v \right)\mathrm{d}u =$$

$$f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \int_0^x \int_0^u (u-v)f'''(v)\,\mathrm{d}v\,\mathrm{d}u,$$

$$\int_0^x \int_0^u (u-v)f'''(v)\,\mathrm{d}v\,\mathrm{d}u = \int_0^x \int_v^x (u-v)f'''(v)\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\int_0^x f'''(v)(x-v)^2\,\mathrm{d}v$$

(b) Apply part (a) with $f(x) = e^{ix}$.

(c) Use integration by parts and the fact that $\frac{\mathrm{d}}{\mathrm{d}u}\frac{e^{iu}-1}{i} = e^{iu}$ to obtain

$$\frac{i^3}{2}\int_0^x (x-u)^2 e^{iu}\,\mathrm{d}u = \frac{i^3}{2}\left[(x-u)^2 \frac{e^{iu}-1}{i} \right]_{u=0}^{u=x} - \frac{i^3}{2}\int_0^x (-2)(x-u)\frac{e^{iu}-1}{i}\,\mathrm{d}u =$$

$$0 - \int_0^x (x-u)(e^{iu}-1)\,\mathrm{d}u$$

(d) $\left| \frac{i^3}{2}\int_0^x (x-u)^2 e^{iu}\,\mathrm{d}u \right| \leq \frac{1}{2}\int_0^x |x-u|^2\,\mathrm{d}u = \frac{1}{6}|x|^3$, $\left| -\int_0^x (x-u)(e^{iu}-1)\,\mathrm{d}u \right| \leq \int_0^x 2|x-u|\,\mathrm{d}u = |x|^2$

(e) $\left| \varphi(t) - 1 - it\mathbb{E}(X) + \frac{t^2\mathbb{E}(X^2)}{2} \right| = \left| \mathbb{E}\left(e^{itX} - \sum_{k=0}^{2}\frac{(itX)^k}{k!} \right) \right| \leq \mathbb{E}\left(\left| e^{itX} - \sum_{k=0}^{2}\frac{(itX)^k}{k!} \right| \right) \overset{(d)}{\leq}$ $\mathbb{E}\left(\min\left\{ \frac{|tX|^3}{6},\, |tX|^2 \right\} \right)$

Remark: $\lim_{t\to 0}\frac{1}{|t|^2}\mathbb{E}\left(\min\left\{ \frac{|t|^3}{6}|X|^3,\, t^2|X|^2 \right\} \right) = \lim_{t\to 0}\mathbb{E}\left(\min\left\{ \frac{|t|}{6}|X|^3,\, |X|^2 \right\} \right) = 0$ by the dominated convergence theorem (we choose X^2 to be the dominating random variable). In other words, we have $\mathbb{E}\left(\min\left\{ \frac{|t|^3}{6}|X|^3,\, t^2|X|^2 \right\} \right) = \bar{\bar{o}}(t^2)$. Thus by (e), we indeed have

$$\varphi(t) = 1 + i\mathbb{E}(X)t - \frac{1}{2}\mathbb{E}(X^2)t^2 + \bar{\bar{o}}(t^2),$$

as we stated in the proof of the CLT on page 94 of the scanned lecture notes.