Limit/large dev. thms. HW assignment 5. Due Wednesday, March 20.

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. Let T_1 denote the first time when the one dimensional simple symmetric random walk (X_n) reaches level 1 (see page 59 of the scanned lecture notes).

   (a) Show that $\mathbb{P}(T_1 > n) = \mathbb{P}(X_n = 0) + \mathbb{P}(X_n = 1)$.
   Hint: Use the reflection principle (see page 58).

   (b) Use the result of an earlier homework to show that

   $$\lim_{n\to\infty} \frac{\mathbb{P}(T_1 > n)}{\sqrt{\frac{2}{\pi}\frac{1}{\sqrt{n}}}} = 1.$$

   (c) Show that $\mathbb{E}(T_1) = +\infty$.

2. Let $y_{n,j} \in \mathbb{R}$, $j = 1, 2, \ldots, N_n$, $n = 1, 2, \ldots$ and assume

   $$\lim_{n\to\infty} \max_{1\le j\le N_n} |y_{n,j}| = 0, \qquad \sup_n \sum_{j=1}^{N_n} |y_{n,j}| < \infty, \qquad \lim_{n\to\infty} \sum_{j=1}^{N_n} y_{n,j} = y.$$

   Prove

   $$\lim_{n\to\infty} \prod_{j=1}^{N_n} (1 + y_{n,j}) = e^y.$$

   Hint: Use the first order Taylor expansion of the logarithm function: if $|y| < \frac{1}{2}$ then $|\ln(1+y) - y| \le Cy^2$.

3. The classical birthday paradox is the fact that if we choose 23 people randomly, then with probability at least 1/2 there will be at least two of them who celebrate their birthdays on the same day of the year. This fact can be viewed as the $n = 365$ case of the following limit theorem.

   Let us fix $n \in \mathbb{N}$ and let $X_{n,j}$, $j = 1, 2, \ldots$ be i.i.d. random variables uniformly distributed on $\{1, 2, \ldots, n\}$. Define

   $$T_n := \min\{\, k \,:\, \exists j < k,\ X_{n,j} = X_{n,k} \,\}.$$

   In plain words: T_n is the index j when the first coincidence of the values is observed. Note that by the pigeonhole principle, we have $\mathbb{P}(T_n \le n + 1) = 1$.

   Prove that $T_n/\sqrt{n}$ converges weakly as $n \to \infty$ and identify the limiting distribution. More specifically, prove that for any $x \ge 0$,

   $$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n}{\sqrt{n}} > x\right) = e^{-x^2/2}.$$

   Hint: Use the result of exercise 2.