Limit/large dev. thms. HW assignment 3. Solutions

1. (a) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := X_1 + X_2$, where X_1 and X_2 are i.i.d. copies of X. Find the Legendre transform of the logarithmic moment generating function of Y.

   (b) Let I denote the Legendre transform of the logarithmic moment generating function of X. Let $Y := aX + b$ (where $a, b \in \mathbb{R}$). Find the Legendre transform of the log. mom. gen. function of Y.

   (c) Let $Y_1, Y_2, \ldots$ denote i.i.d. integer-valued random variables with distribution

$$\mathbb{P}(Y_i = -2k) = 2^{-(k+1)}, \qquad k = 0, 1, 2, \ldots \tag{1}$$

   Find $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(Y_1 + \cdots + Y_n \le nx)\right)$ for any $x \in \mathbb{R}$.

   (d) Let $Y_1, Y_2, \ldots$ denote i.i.d. integer-valued random variables with distribution

$$\mathbb{P}(Y_i = -1) = 1/4, \quad \mathbb{P}(Y_i = 0) = 1/2, \quad \mathbb{P}(Y_i = 1) = 1/4. \tag{2}$$

   Find $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left(Y_1 + \cdots + Y_n \le nx\right)\right)$ for any $x \in \mathbb{R}$.

Solution:

(a) Let $\widehat{I}_X(\lambda) := \ln(\mathbb{E}[e^{\lambda X}])$ and $\widehat{I}_Y(\lambda) := \ln(\mathbb{E}[e^{\lambda Y}]) = \ln(\mathbb{E}[e^{\lambda X}]^2) = 2\ln(\mathbb{E}[e^{\lambda X}]) = 2\widehat{I}_X(\lambda)$.
We have $I_X(x) = \sup_\lambda\{\lambda x - \widehat{I}_X(\lambda)\}$, thus we obtain

$$I_Y(x) := \sup_\lambda\{\lambda x - \widehat{I}_Y(\lambda)\} = \sup_\lambda\{\lambda x - 2\widehat{I}_X(\lambda)\} = 2\sup_\lambda\{\lambda \frac{x}{2} - \widehat{I}_X(\lambda)\} = 2I_X(\frac{x}{2}).$$

Remark: Let $X_1, X_2, \ldots$ be i.i.d. with the same distribution as X. Let $Y_1, Y_2, \ldots$ be i.i.d. with the same distribution as Y. By our assumption $X_1 + \cdots + X_{2n}$ has the same distribution as $Y_1 + \cdots + Y_n$. Thus, applying the heuristic version of Cramér's theorem twice, we obtain

$$e^{-nI_Y(x)} \approxeq \mathbb{P}(\frac{Y_1 + \cdots + Y_n}{n} \approx x) = \mathbb{P}(\frac{X_1 + \cdots + X_{2n}}{n} \approx x) \approx \mathbb{P}(\frac{X_1 + \cdots + X_{2n}}{2n} \approx \frac{x}{2}) \approxeq e^{-2nI_X(\frac{x}{2})},$$

which is another (heuristic) way of seeing $I_Y(x) = 2I_X(\frac{x}{2})$.

(b) Let $\widehat{I}_X(\lambda) := \ln(\mathbb{E}[e^{\lambda X}])$ and $\widehat{I}_Y(\lambda) := \ln(\mathbb{E}[e^{\lambda Y}]) = \widehat{I}_X(a\lambda) + \lambda b$ by HW1.1(a).
We have $I_X(x) = \sup_\lambda\{\lambda x - \widehat{I}_X(\lambda)\}$, thus we obtain

$$I_Y(x) := \sup_\lambda\{\lambda x - \widehat{I}_Y(\lambda)\} = \sup_\lambda\{\lambda(x-b) - \widehat{I}_X(a\lambda)\} =$$
$$\sup_\lambda\{a\lambda\frac{x-b}{a} - \widehat{I}_X(a\lambda)\} = \sup_{\lambda'}\{\lambda'\frac{x-b}{a} - \widehat{I}_X(\lambda')\} = I_X\left(\frac{x-b}{a}\right).$$

(c) Let X denote an (optimistic) GEO($\frac{1}{2}$) random variable. Then $X-1$ is a pessimistic GEO($\frac{1}{2}$) random variable and $-2(X-1)$ has the same distribution as Y_i in equation (1) above. We know from class (see page 28 of scanned lecture notes) that $I_X(x) = (x-1)\ln\left(\frac{x-1}{1-\frac{1}{2}}\right) - x\cdot\ln(x) - \ln(\frac{1}{2})$ and if $Y = -2(X-1) = -2X+2$ then we obtain $I_Y(x) = I_X(\frac{x-2}{-2}) = I_X(1-\frac{1}{2}x)$ by part (b) of this exercise. Also, it is well known that $\mathbb{E}(X) = 1/(1/2) = 2$, thus $\mathbb{E}(Y) = -2(2-1) = -2$. Thus by Carmér's theorem we obtain

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\mathbb{P}(Y_1 + \cdots + Y_n \le nx)\right) = -\min_{y\in(-\infty,x]} I_Y(y) = \begin{cases} 0 & \text{if } x \ge \mathbb{E}(Y), \\ -I_Y(x) & \text{if } x \le \mathbb{E}(Y). \end{cases} \tag{3}$$

(d) Let X_1 and X_2 denote i.i.d. BER(1/2) random variables. Let $Z := X_1 + X_2$ and $Y := Z - 1$. Then Y has the same distribution as Y_i from (2). We know from page 8 of the scanned lecture notes that $I_X(x) = (1-x)\ln\left(\frac{1-1/2}{1-x}\right) + x\ln\left(\frac{1/2}{x}\right)$. Now by part (a) of this exercise we have $I_Z(x) = 2I_X(x/2)$ and by part (b) of this exercise we have $I_Y(x) = I_Z(x+1)$, thus $I_Y(x) = 2I_X(\frac{x+1}{2})$. Also $\mathbb{E}(Y) = 0$ and again by Carmér's theorem we obtain (3).

2. Let $X_1, X_2, \dots$ denote i.i.d. random variables with EXP(λ) distribution, i.e., the density function of X_i is $f(x) = \lambda e^{-\lambda x} \mathbb{1}[x \geq 0]$. Let $S_n = X_1 + \cdots + X_n$.

(a) Use induction on n to show that the density function of S_n is

$$f_n(x) = \lambda^n e^{-\lambda x} \frac{x^{n-1}}{(n-1)!} \mathbb{1}[x \geq 0].$$

Hint: Use the convolution formula stated on page 20 of the scanned lecture notes.

(b) Calculate the logarithmic moment generating function $\mu \mapsto \widehat{I}(\mu)$ of X_i. For which values of μ do we have $\widehat{I}(\mu) < +\infty$?

(c) Calculate the Legendre transform $I(x)$ of $\widehat{I}(\mu)$. For which values of x do we have $\widehat{I}(x) < +\infty$?

(d) Give a formula for $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(S_n/n \geq x)\right)$ for any $x \geq 1/\lambda$ using Cramér's theorem (see page 21 of scanned lecture notes).

(e) Calculate $\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(S_n/n \geq x)\right)$ directly using the formula for the density function f_n of S_n. *Hint:* Use Laplace's principle (similarly to page 15 of the scanned lecture notes) and the crude Stirling formula (see page 3 of scanned):

$$n^n e^{1-n} \leq n! \leq (n+1)^{n+1} e^{-n} \tag{4}$$

Solution:

(a) The induction hypothesis holds if $n = 1$: we indeed have $f_1(x) = f(x) = \lambda^1 e^{-\lambda x} \frac{x^0}{0!} \mathbb{1}[x \geq 0]$. Now let's show that if we assume that it holds for n then it also holds for $n+1$: for any $x \geq 0$ we have

$$f_{n+1}(x) = (f_n * f)(x) = \int_{-\infty}^{\infty} f_n(y) f(x-y)\,\mathrm{d}y =$$

$$\int_{-\infty}^{\infty} \lambda^n e^{-\lambda y} \frac{y^{n-1}}{(n-1)!} \mathbb{1}[y \geq 0] \cdot \lambda e^{-\lambda(x-y)} \mathbb{1}[x-y \geq 0]\,\mathrm{d}y = \lambda^{n+1} e^{-\lambda x} \int_0^x \frac{y^{n-1}}{(n-1)!}\,\mathrm{d}y = \lambda^{n+1} e^{-\lambda x} \frac{x^n}{n!}$$

(b) Since the notation λ is already reserved for the parameter of EXP(λ), let us denote by μ the variable of the logarithmic moment generating function $\widehat{I}(\mu) = \ln(\mathbb{E}(e^{\mu X_i}))$.

$$\mathbb{E}(e^{\mu X_i}) = \int_0^{\infty} e^{\mu x} f(x)\,\mathrm{d}x = \int_0^{\infty} \lambda e^{(\mu-\lambda)x}\,\mathrm{d}x = \frac{\lambda}{\lambda - \mu}, \quad \text{if} \quad \mu < \lambda,$$

thus $\widehat{I}(\mu) = \ln\left(\frac{\lambda}{\lambda-\mu}\right) = \ln(\lambda) - \ln(\lambda - \mu)$ if $\mu < \lambda$ and $\widehat{I}(\mu) = +\infty$ if $\mu \geq \lambda$.

(c) $I(x) = \sup_\mu \{\mu x - \widehat{I}(\mu)\}$. $\widehat{I}'(\mu) = \frac{1}{\lambda - \mu}$. Given x we want μ^* such that $\widehat{I}'(\mu^*) = x$. Thus $\mu^* = \lambda - \frac{1}{x}$ and if $x > 0$ then

$$I(x) = \mu^* x - \widehat{I}(\mu^*) = \lambda x - 1 - \ln\left(\frac{\lambda}{\lambda - \mu^*}\right) = \lambda x - 1 - \ln(\lambda x).$$

If $x < 0$ then $I(x) = +\infty$ because for $\mu \leq 0$ we have $\widehat{I}(\mu) \leq 0$, thus $\lim_{\mu \to -\infty}\{\mu x - \widehat{I}(\mu)\} = +\infty$.

(d) We have $\mathbb{E}(X_i) = 1/\lambda$, thus by Cramér's theorem we have

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}(S_n/n \geq x)\right) = -\inf_{y \geq x} I(y) = -I(x) = \ln(\lambda x) + 1 - \lambda x, \qquad x > 1/\lambda. \tag{5}$$

(e) The density function of S_n/n is $g_n(x) = n f_n(nx)$, thus for $x > 1/\lambda$ we have

$$g_n(x) = n\lambda^n e^{-\lambda n x} \frac{(nx)^{n-1}}{(n-1)!} = n\lambda^n e^{-\lambda n x} \frac{n^n x^{n-1}}{n!}, \qquad \mathbb{P}(S_n/n \geq x) = \int_x^{\infty} g_n(y)\,\mathrm{d}y,$$

therefore by (4) we get

$$n\left(\frac{n}{n+1}\right)^n \int_x^{\infty} \frac{1}{y}\left(\lambda e^{-\lambda y} e y\right)^n \mathrm{d}y \leq \mathbb{P}(S_n/n \geq x) \leq n \int_x^{\infty} \frac{1}{y}\left(\lambda e^{-\lambda y} e y\right)^n \mathrm{d}y,$$

thus

$$\mathbb{P}(S_n/n \geq x) \approxeq \int_x^\infty \frac{1}{y}(\lambda e^{-\lambda y} e y)^n \, \mathrm{d}y = \int_x^\infty \frac{1}{y} e^{n(\ln(\lambda y) - \lambda y + 1)} \, \mathrm{d}y = \int_x^\infty \frac{1}{y} e^{-nI(y)} \, \mathrm{d}y. \tag{6}$$

We will show

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\int_x^\infty \frac{1}{y} e^{-nI(y)} \, \mathrm{d}y \right) = -I(x). \tag{7}$$

We have

$$\frac{1}{2x} \int_x^{2x} e^{-nI(y)} \, \mathrm{d}y \leq \int_x^\infty \frac{1}{y} e^{-nI(y)} \, \mathrm{d}y \leq \frac{1}{x} \int_x^\infty e^{-nI(y)} \, \mathrm{d}y \tag{8}$$

Applying the Laplace lemma twice, for any $x > 1/\lambda$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\int_x^\infty e^{-nI(y)} \, \mathrm{d}y \right) = -\inf_{y \geq x} I(y) = -I(x),$$

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\int_x^{2x} e^{-nI(y)} \, \mathrm{d}y \right) = -\inf_{y \in [x, 2x]} I(y) = -I(x).$$

Putting these together with (8), we obtain (7), which, together with (6) gives an alternative proof of (5).

3. Let X_n denote an optimistic geometric random variable with success probability $p = 1/n$. Show that $X_n/\mathbb{E}(X_n)$ converges in distribution as $n \to \infty$ and identify the limiting distribution.

Solution: For any non-negative integer m we have $\mathbb{P}(X_n > m) = (1-p)^m$ since $X_n > m$ means the first m trials were unsuccessful. Note that $\mathbb{E}(X_n) = \frac{1}{p} = n$. For any $x \geq 0$ we have

$$\mathbb{P}\left(\frac{X_n}{\mathbb{E}(X_n)} \leq x\right) = \mathbb{P}\left(\frac{X_n}{n} \leq x\right) = \mathbb{P}\left(X_n \leq xn\right) = \mathbb{P}\left(X_n \leq \lfloor xn \rfloor\right) =$$
$$1 - \mathbb{P}\left(X_n > \lfloor xn \rfloor\right) = 1 - (1-p)^{\lfloor xn \rfloor} = 1 - \left(1 - \frac{1}{n}\right)^{\lfloor xn \rfloor} \to 1 - e^{-x}, \qquad n \to \infty.$$

For any $x \leq 0$ we have $\mathbb{P}\left(\frac{X_n}{\mathbb{E}(X_n)} \leq x\right) = 0$. Thus, if we denote by F_n the c.d.f. of $X_n/\mathbb{E}(X_n)$ and we denote $F(x) = 1 - e^{-x}$ for $x \geq 0$ and $F(x) = 0$ for $x \leq 0$ then F_n converges point-wise to F.
This means $\frac{X_n}{\mathbb{E}(X_n)} \Rightarrow \mathrm{EXP}(1)$, since F is the c.d.f. of the EXP(1) distribution.