Limit/large dev. thms. HW assignment 2. Due Wednesday, Mar. 6 at 12.15pm

Note: Each of the 3 questions is worth 10 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises with a horizontal line. Highlight the final answer. If you submit your homework electronically, pdf format is preferred.

1. Let $\lambda \mapsto Z(\lambda)$ denote the moment generating function of the random variable X. If $Z(\mu) < +\infty$, let $X^{(\mu)}$ denote the *exponentially tilted* random variable. The cumulative distribution function of $X^{(\mu)}$ is

$$F_\mu(x) = \mathbb{P}(X^{(\mu)} \le x) = \frac{\mathbb{E}(e^{\mu X} \mathbb{1}[X \le x])}{Z(\mu)}.$$

   We have learnt some facts about exponential tilting in class, see page 17-18 of scanned lecture notes.

   (a) Show that $X^{(\lambda)^{(\mu)}} \sim X^{(\lambda+\mu)}$. In words: tilting the tilted random variable amounts to tilting the original random variable with the sum of the two tiltings.

   (b) Express the logarithmic moment generating function $\widehat{I}_\mu$ of $X^{(\mu)}$ using the logarithmic moment generating function $\widehat{I}$ of X.

   (c) Express the Legendre transform I_μ of $\widehat{I}_\mu$ using the Legendre transform I of $\widehat{I}$.

2. In this exercise $X^{(\mu)}$ denotes the random variable that we obtain by exponentially tilting the distribution of the random variable X.

   (a) Show that if $X \sim \mathrm{BIN}(n,p)$ then $X^{(\mu)} \sim \mathrm{BIN}(n,p')$ for some $p' = p'(p,\mu)$ and that any $p' \in (0,1)$ can be obtained by choosing $\mu \in \mathbb{R}$ appropriately.

   (b) Show that if $X \sim \mathrm{POI}(\lambda)$ then $X^{(\mu)} \sim \mathrm{POI}(\lambda')$ for some $\lambda' = \lambda'(\lambda,\mu)$ and that any $\lambda' \in (0,+\infty)$ can be obtained by choosing $\mu \in \mathbb{R}$ appropriately.

   (c) If $X \sim \mathrm{EXP}(\lambda)$, find the values of μ for which $Z(\mu) < +\infty$ (i.e., find the domain of the moment generating function $Z(\cdot)$). Show that $X^{(\mu)} \sim \mathrm{EXP}(\lambda')$ for some $\lambda' = \lambda'(\lambda,\mu)$ and that any $\lambda' \in (0,+\infty)$ can be obtained by choosing μ from the domain of $Z(\cdot)$ appropriately.

   (d) Show that if $X \sim \mathcal{N}(m,\sigma^2)$ then $X^{(\mu)} \sim \mathcal{N}(m',\sigma^2)$ for some $m' = m'(m,\mu,\sigma)$ and that any $m' \in (-\infty,+\infty)$ can be obtained by choosing $\mu \in \mathbb{R}$ appropriately.

3. Let $X_1, X_2, \dots$ denote i.i.d. non-negative integer-valued random variables with distribution $\mathbb{P}(X_i = k) = p_k$, where $k = 0,1,2,\dots$ Let $\lambda \in \mathbb{R}$ such that $Z(\lambda) = \mathbb{E}[e^{\lambda X_i}] < +\infty$. Let $S_n = X_1 + \dots + X_n$.

   Let $X_1^{(\lambda)}, X_2^{(\lambda)}, \dots$ denote i.i.d. non-negative integer-valued random variables with distribution

$$\mathbb{P}(X_i^{(\lambda)} = k) = \frac{1}{Z(\lambda)} e^{\lambda k} p_k, \quad \text{where} \quad k = 0,1,2,\dots$$

   (a) Show that we have $\mathbb{P}(X_1^{(\lambda)} + \dots + X_n^{(\lambda)} = k) = \frac{e^{\lambda k}\mathbb{P}(X_1+\dots+X_n=k)}{Z(\lambda)^n}$.
   Instruction: This could be easily derived from the Lemma on page 20 of the scanned lecture notes, but since we only proved that lemma in the absolutely continuous case, I ask you to write down a complete proof of this sub-exercise only using the basic facts about exponential tilting (page 17-18 of scanned lecture notes).

   (b) Show that $\mathbb{P}(X_1^{(\lambda)} = k \mid X_1^{(\lambda)} + \dots + X_n^{(\lambda)} = m) = \mathbb{P}(X_1 = k \mid X_1 + \dots + X_n = m)$.

   (c) If $X_i \sim \mathrm{POI}(\mu)$, what is the conditional distribution of X_1 given that $X_1 + \dots + X_n = \lfloor nx \rfloor$?
   Hint: This conditional distribution will turn out to be a famous distribution. In your proof you will need to use that the sum of independent Poisson random variables is a Poisson random variable.

   (d) If $X_i \sim \mathrm{POI}(\mu)$, show that for any $x > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}(X_1 = k \mid X_1 + \dots + X_n = \lfloor nx \rfloor) = e^{-x}\frac{x^k}{k!}.$$

   Remark: This last result is a rigorous version of the result proved heuristically on page 25 of the scanned lecture notes (also note that an exponentially tilted Poisson random variable is still a Poisson random variable with a different parameter according to exercise 2(b) above)