Limit / large dev. thms. Exercise, lecture 15

1. Let $a \in \mathbb{R}_+$. Let $X_n \sim \mathrm{BIN}(n, \frac{a}{n})$. Let $X \sim \mathrm{POI}(a)$.

   (a) Find the characteristic function φ_n of X_n and the characteristic function φ of X.

   (b) Prove that $X_n \Rightarrow X$ by showing that φ_n converges point-wise to φ.

2. For any $z \in (0, +\infty)$, we define the Gamma function $\Gamma(z)$ by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x.$$

   Note that $\Gamma(z)$ can by extended analytically to all complex numbers except the non-positive integers. The *Weierstrass-identity* is the following formula:

$$\Gamma(z+1) = e^{-\gamma \cdot z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \cdot e^{\frac{z}{n}},$$

   where γ is the Euler-constant:

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log(n).$$

   The goal of this exercise is to prove the Weierstrass-identity using probabilistic methods.

   (a) Let $X_1, \dots, X_n$ denote i.i.d. $EXP(1)$ random variables, moreover let $Y_1, \dots, Y_n$ denote independent random variables, $Y_k \sim EXP(k)$. Use the memoryless property of exponential distribution to show that

$$M_n \sim T_n, \qquad \text{where} \qquad M_n := \max\{X_1, \dots, X_n\}, \qquad T_n := Y_1 + \dots + Y_n.$$

   (b) Recall that $M_n - \log(n)$ converges in distribution as $n \to \infty$ to the standard Gumbel distribution, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\left(M_n - \ln(n) \le x\right) = \exp\left(-e^{-x}\right), \qquad x \in \mathbb{R}$$

   (c) Calculate the moment generating function $Z(\lambda)$ of the standard Gumbel distribution, i.e., show

$$Z(\lambda) = \Gamma(1-\lambda)$$

   (d) Let $Z_n = Y_n - \frac{1}{n}$. Show that the sum $Z = \sum_{n=1}^{\infty} Z_n$ is well-defined by showing that it has zero expectation and finite variance. Show that $Z + \gamma$ has standard Gumbel distribution.

   (e) Use characteristic functions to show that

$$\Gamma(1 - it) = e^{it\gamma} \prod_{n=1}^{\infty} \left(1 - \frac{it}{n}\right)^{-1} \cdot e^{-it/n},$$

   i.e., show that the Weierstrass identity holds for $z = -it$.

Lemma. Let $X_1, \dots, X_n$ denote i.i.d. $EXP(1)$ random variables, moreover let $Y_1, \dots, Y_n$ denote independent random variables, $Y_k \sim EXP(k)$. We have

$$M_n \sim T_n, \qquad \text{where} \qquad M_n := \max\{X_1, \dots, X_n\}, \qquad T_n := Y_1 + \dots + Y_n.$$

Proof. Imagine that we have n i.i.d. clocks and X_i is the time when clock with index i rings, $1 \le i \le n$.
M_n is the time when the last clock rings.
Let $T_0 = 0$ and let us denote by T_k the time when you hear the k'th ring.
For example we have $T_1 = \min\{X_1, \dots, X_n\}$ and $T_n = M_n$.
Let us denote by $\hat{Y}_k = T_k - T_{k-1}$ the time that elapses between T_{k-1} and T_k.
We have

$$M_n = \hat{Y}_1 + \dots + \hat{Y}_n.$$

We will show that $\hat{Y}_1, \dots, \hat{Y}_n$ are independent and $\hat{Y}_k \sim \text{EXP}(n-k+1)$.
What is the distribution of the time when you hear the first clock ring?
It is known that

$$\hat{Y}_1 = \min\{X_1, \dots, X_n\} \sim \text{EXP}(n) \sim Y_n$$

Then, by the memoryless property, the remaining $n-1$ clocks start afresh, so after the first ring, we still have $n-1$ i.i.d. EXP(1) clocks.

Then we can repeat the above argument inductively: the gap $\hat{Y}_2$ between the first and the second ring has $\text{EXP}(n-1)$ distribution, just like Y_{n-1}, etc.

Finally the gap $\hat{Y}_n$ between ring $n-1$ and ring n has EXP(1) distribution, since there is only one EXP(1) clock left.