EX: $\boxed{\mathrm{BIN}\left(n, \frac{\lambda}{n}\right) \Rightarrow \mathrm{POI}(\lambda)}$ AS $n \to \infty$

PROOF: LET $X_n \sim \mathrm{BIN}\left(n, \frac{\lambda}{n}\right)$, $X \sim \mathrm{POI}(\lambda)$

$$\lim_{n\to\infty} \underbrace{\binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}}_{\mathbb{P}(X_n = k)} = \underbrace{e^{-\lambda} \cdot \frac{\lambda^k}{k!}}_{\mathbb{P}(X = k)}, \quad k \in \mathbb{N}$$

THUS BY CLAIM STATED ON PAGE 44,

WE HAVE $X_n \Rightarrow X$ ✓

CENTRAL LIMIT THEOREM (C.L.T.):

$X_1, X_2, \ldots$ I.I.D. $\mathbb{E}(X_i) = m$, $\mathrm{Var}(X_i) = \sigma^2 < +\infty$

$S_n = X_1 + \ldots + X_n$, THEN

$$\boxed{\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \Longrightarrow \mathcal{N}(0,1)}$$

PROOF: IN A FEW WEEKS

$$\boxed{\varphi(x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}$$

THAT IS: $\forall x \in \mathbb{R}$:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n \cdot m}{\sigma \cdot \sqrt{n}} \le x\right) = \Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy$$

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WE WILL NOW PROVE A SPECIAL CASE: WHEN $\boxed{X_i \sim EXP(1)}$. AT THE SAME TIME, WE WILL PROVE STIRLING'S FORMULA:

$$\boxed{\lim_{n\to\infty} \frac{\sqrt{2\pi}\cdot n^{n+\frac{1}{2}}\cdot e^{-n}}{n!} = 1}$$

NOTE: $\mathbb{E}(X_i) = 1$, $Var(X_i) = 1$

$\mathbb{E}(S_n) = n$, $Var(S_n) = n$

WANT: $\boxed{\frac{S_n - n}{\sqrt{n}} \Rightarrow \mathcal{N}(0,1)}$

HW 3.3 (a)

KNOW: P.D.F. OF S_n : $f_n(x) \overset{A}{=} e^{-x}\cdot \frac{x^{n-1}}{(n-1)!}\cdot \mathbb{1}[x \geq 0]$

THUS: P.D.F. OF $(S_n - n)/\sqrt{n}$ IS:

$$g_n(x) \overset{D}{=} \sqrt{n}\cdot f_n(n+\sqrt{n}\cdot x) =$$

$$= \sqrt{n}\cdot e^{-(n+\sqrt{n}\cdot x)}\cdot \frac{(n+\sqrt{n}\cdot x)^{n-1}}{(n-1)!}\cdot \mathbb{1}[n+\sqrt{n}\cdot x \geq 0]$$

$$\overset{B}{=} \frac{n^{3/2}\cdot e^{-(n+\sqrt{n}\cdot x)}\cdot (n+\sqrt{n}\cdot x)^{n-1}}{n!}\cdot \mathbb{1}[x \geq -\sqrt{n}]$$

DEF: LET $\boxed{\tilde{g}_n(x) := g_n(x) \cdot \frac{n!}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}}}$

C

LEMMA: $\forall\, K \in \mathbb{R}_+$ $\boxed{\tilde{g}_n(x) \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

AS $n \to \infty$, UNIFORMLY FOR ALL $x \in [-K, K]$

PROOF: $\tilde{g}_n(x) = \frac{n^{3/2} \cdot e^{-(n+\sqrt{n}\cdot x)} \cdot (n+\sqrt{n}\cdot x)^{n-1}}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}} =$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-(n+\sqrt{n}\cdot x)}}{e^{-n}} \cdot \frac{(n+\sqrt{n}\cdot x)^{n-1}}{n^{n-1}} =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\sqrt{n}\cdot x} \cdot \left(1 + \frac{x}{\sqrt{n}}\right)^{n-1} =$$

$$= \frac{\left(1+\frac{x}{\sqrt{n}}\right)^{-1}}{\sqrt{2\pi}} \cdot \left(e^{-x/\sqrt{n}} \cdot \left(1 + \frac{x}{\sqrt{n}}\right)\right)^{n}$$

IT REMAINS TO SHOW:

$$\ln\left(\lim_{n\to\infty}\left(e^{-x/\sqrt{n}} \cdot \left(1+\frac{x}{\sqrt{n}}\right)\right)^{n}\right) = \lim_{n\to\infty} n \cdot \left(-\frac{x}{\sqrt{n}} + \ln\left(1+\frac{x}{\sqrt{n}}\right)\right)$$

$$= \lim_{n\to\infty} n \cdot \left(-\frac{x}{\sqrt{n}} + \frac{x}{\sqrt{n}} - \frac{1}{2}\frac{x^2}{n}\right) = -\frac{1}{2}x^2 \quad \checkmark$$

$\boxed{\ln(1+y) = y - \frac{1}{2}y^2 + O(y^3)}$

LET $\boxed{\alpha_n := \dfrac{n!}{\sqrt{2\pi}\cdot n^{n+1/2}\cdot e^{-n}}}$

NOTE:

E

$$\boxed{\tilde{g}_n(x) = g_n(x)\cdot \alpha_n}$$

THM (DE MOIVRE, STIRLING): $\boxed{\lim\limits_{n\to\infty} \alpha_n = 1}$

F

PROOF: ENOUGH TO SHOW THAT $\forall \varepsilon > 0$:

$$\boxed{1-\varepsilon \leq \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n \leq \frac{1}{1-\varepsilon}}$$

LET $K := \frac{1}{\sqrt{\varepsilon}}$ AND NOTE: $\boxed{\text{CHEBYSHEV}}$

$$\int_{-K}^{K} g_n(x)\,dx = \mathbb{P}\left(\left|\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\operatorname{Var}(S_n)}}\right| \leq K\right) \geq 1 - \frac{1}{K^2} = 1-\varepsilon$$

$$\int_{-K}^{K} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \overset{\text{CHEBYSHEV}}{\geq} 1-\varepsilon \quad , \text{ THUS}$$

$$\alpha_n\cdot(1-\varepsilon) \leq \alpha_n \cdot \int_{-K}^{K} g_n(x)\,dx = \int_{-K}^{K} \tilde{g}_n(x)\,dx \longrightarrow \int_{-K}^{K} \varphi(x)\,dx \leq 1$$

THUS $\limsup\limits_{n\to\infty} \alpha_n \leq \frac{1}{1-\varepsilon}$, AND

$$\alpha_n = \alpha_n \cdot \int_{-\infty}^{\infty} g_n(x)\,dx = \int_{-\infty}^{\infty} \tilde{g}_n(x)\,dx \geq \int_{-K}^{K} \tilde{g}_n(x)\,dx \longrightarrow \underbrace{\int_{-K}^{K} \varphi(x)\,dx}_{\geq 1-\varepsilon}$$

THUS $\liminf\limits_{n\to\infty} \alpha_n \geq 1-\varepsilon$ ✓

THUS $\boxed{g_n(x) \to \varphi(x)}$ (POINTWISE) $\forall x \in \mathbb{R}$.

IN ORDER TO COMPLETE THE PROOF OF C.L.T. FOR EXP(1) SUMMANDS, WE NEED:

<u>LEMMA</u> (SCHEFFÉ): IF f_n IS A P.D.F. FOR EACH n AND f IS A P.D.F. AND IF $\boxed{\lim_{n\to\infty} f_n(x) = f(x)}$ (POINTWISE) $\forall x \in \mathbb{R}$, THEN

(A) $\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = 0$ $\left(f_n \xrightarrow{L_1} f \right)$

(B) $\boxed{F_n \Rightarrow F}$ $F_n(x) = \int_{-\infty}^{x} f_n(y)\,dy$, $F(x) = \int_{-\infty}^{x} f(y)\,dy$

<u>PROOF</u>: (A) IDEA: USE $\boxed{|f_n - f| \overset{G}{=} f_n + f - 2\cdot(f \wedge f_n)}$

AND DOMINATED CONVERGENCE.

$0 \le f(x) \wedge f_n(x) \le f(x)$ AND $\int_{-\infty}^{\infty} f(x)\,dx = 1$

MOREOVER $\lim_{n\to\infty} f(x) \wedge f_n(x) = f(x)$, THUS

$\boxed{\lim_{n\to\infty} \int_{-\infty}^{\infty} (f_n(x) \wedge f(x))\,dx = 1}$ BY DOMINATED CONVERGENCE.

$$\int_{-\infty}^{\infty} |f_n(x) - f(x)|\,dx = \underbrace{\int_{-\infty}^{\infty} f_n(x)\,dx}_{1} + \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1} - 2 \cdot \underbrace{\int_{-\infty}^{\infty} (f_n(x) \wedge f(x))\,dx}_{\xrightarrow[n \to \infty]{} 1}$$

THUS $\lim\limits_{n \to \infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\,dx = 2 - 2 = 0$ ✓

Ⓑ $$|F_n(x) - F(x)| = \left| \int_{-\infty}^{x} (f_n(x) - f(x))\,dx \right| \leq$$

$$\int_{-\infty}^{x} |f_n(x) - f(x)|\,dx \leq \int_{-\infty}^{\infty} |f_n(x) - f(x)|\,dx \xrightarrow[n \to \infty]{} 0$$

<u>THM</u> (CRAMÉR-SLUTSKY)

DETERMINISTIC CONSTANT

IF $\boxed{X_n \Rightarrow X}$ AND $\boxed{Y_n \Rightarrow c}$ THEN

Ⓐ $X_n + Y_n \Rightarrow X + c$

Ⓑ $X_n \cdot Y_n \Rightarrow X \cdot c$

Ⓒ $X_n / Y_n \Rightarrow X / c$ IF $c \neq 0$.

<u>PROOF</u>: WE WILL ONLY PROVE Ⓐ

SEE NEXT PAGE.

(A) LET $F_n(x) = \mathbb{P}(X_n + Y_n \leq x)$

$$F(x) = \mathbb{P}(X + c \leq x)$$

ENOUGH TO SHOW THAT $\forall x \in \mathbb{R}, \forall \varepsilon > 0$

$$\boxed{F(x-\varepsilon) \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq F(x+\varepsilon)}$$

BECAUSE IF $F(x_-) = F(x)$ THEN THIS WILL IMPLY $\boxed{\lim_{n\to\infty} F_n(x) = F(x)}$

GIVEN x, ε: LET $0 < \delta \leq \varepsilon$: $\boxed{\mathbb{P}(X + c = x - \delta) = 0}$

$$\mathbb{P}(X_n + Y_n \leq x) = \mathbb{P}((X_n + c) + (Y_n - c) \leq x) \geq$$

$$\geq \mathbb{P}((X_n + c) \leq x - \delta, Y_n - c \leq \delta) \geq$$

$$\underbrace{\mathbb{P}(X_n + c \leq x - \delta)}_{\xrightarrow{n\to\infty} \mathbb{P}(X + c \leq x - \delta) = F(x-\delta) \geq F(x-\varepsilon)} - \underbrace{\mathbb{P}(Y_n > c + \delta)}_{\xrightarrow{n\to\infty} 0}$$

THIS SHOWS $\boxed{F(x-\varepsilon) \leq \liminf_{n\to\infty} F_n(x)}$

THE REST OF THE PROOF IS ANALOGOUS AND WE OMIT IT.