CUMULATIVE DISTRIBUTION FUNCTION (C.D.F.):

IF $\mathbb{X}$ IS A RANDOM VARIABLE, LET

$$\boxed{F(x) := \mathbb{P}(\mathbb{X} \leq x)} \leftarrow \text{C.D.F. OF } \mathbb{X}$$

PROPERTIES: $\boxed{\lim_{x \to \infty} F(x) = 1}$, $\boxed{\lim_{x \to -\infty} F(x) = 0}$

NON-DECREASING: $\boxed{x_1 \leq x_2} \Rightarrow \boxed{F(x_1) \leq F(x_2)}$

RIGHT-CONTINUOUS: $\boxed{\lim_{y \to x_+} F(y) = F(x)}$ WHY?

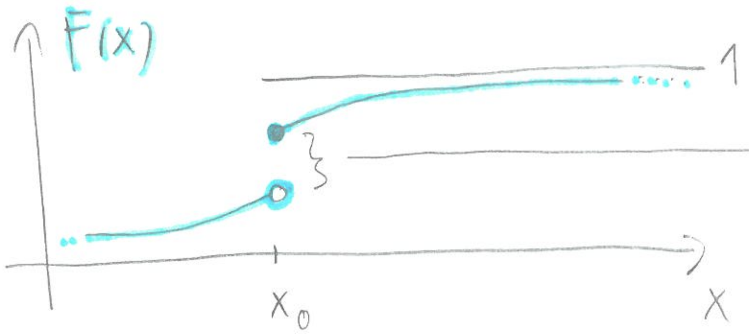
THE EVENTS $\{\mathbb{X} \leq y\}$ DECREASE MONOTONICALLY TO $\{\mathbb{X} \leq x\}$ AS $y \searrow x$, THUS BY THE CONTINUITY OF MEASURE:

$$\boxed{\lim_{y \searrow x} \mathbb{P}(\mathbb{X} \leq y) = \mathbb{P}(\mathbb{X} \leq x)}$$

BUT: $\lim_{y \to x_-} F(y) = F(x_-) = \mathbb{P}(\mathbb{X} < x)$, THUS

$$\boxed{\mathbb{P}(\mathbb{X} = x) = F(x) - F(x_-)}$$

THUS F HAS A DISCONTINUITY AT x_0 IF AND ONLY IF $\mathbb{P}(\mathbb{X} = x_0) > 0$.



$$\boxed{\mathbb{P}(X = x_0) = \text{SIZE OF JUMP}}$$

<u>NOTE</u>: THERE ARE AT MOST COUNTABLY MANY VALUES OF x FOR WHICH $\mathbb{P}(X = x) > 0$, BECAUSE FOR EACH SUCH x, THERE IS $q(x) \in \mathbb{Q}$ SUCH THAT $F(x_-) < q(x) < F(x)$, AND $\mathbb{Q}$ IS COUNTABLE.

<u>DEF</u>: (WEAK CONVERGENCE OF PROBABILITY DISTRIBUTIONS, CONV. IN DISTRIBUTION)

IF F_n IS THE C.D.F. OF X_n
F IS THE C.D.F. OF X

WE SAY THAT $\boxed{F_n \Rightarrow F}$, $\boxed{X_n \Rightarrow X}$ IF

$\boxed{\lim_{n \to \infty} F_n(x) = F(x)}$ FOR ALL $x \in \mathbb{R}$ FOR WHICH $\boxed{\mathbb{P}(X = x) = 0}$

(F_n CONVERGES POINTWISE TO F AT THE POINTS OF CONTINUITY OF F)

NOTE: IF $\mathbb{P}(X_n = x_n) = 1$ (I.E.: X_n IS DETERMINISTIC)

AND $\mathbb{P}(X_\infty = x_\infty) = 1$ THEN $F_n(x) = \begin{cases} 0 & \text{IF } x < x_n \\ 1 & \text{IF } x \geq x_n \end{cases}$

MOREOVER: $\boxed{F_n \Rightarrow F_\infty} \iff \boxed{x_n \to x_\infty}$. PROOF: EASY

NOTE: IF $\boxed{x_n \searrow x_\infty}$ THEN $F_n(x_\infty) = 0$, $F_\infty(x_\infty) = 1$,

$F_n(x_\infty)$ DOES NOT CONVERGE TO $F_\infty(x_\infty)$ AS $n \to \infty$

CLAIM: IF $c \in \mathbb{R}$ THEN $\boxed{X_n \Rightarrow c}$ IF AND

ONLY IF X_n CONVERGES TO c IN PROBABILITY:

$\forall \varepsilon > 0: \lim\limits_{n \to \infty} \mathbb{P}(|X_n - c| \geq \varepsilon) = 0$. PROOF: EASY

THUS WEAK LAW OF LARGE NUMBERS

CAN BE REPHRASED USING THE NOTION

OF WEAK CONVERGENCE:

THM: IF $X_1, X_2, \ldots$ ARE I.I.D., $\mathbb{E}(|X|) < +\infty$

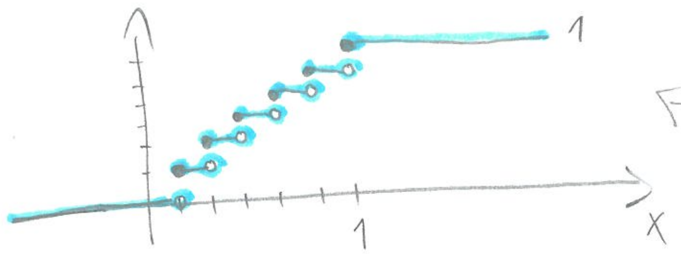
$\mathbb{E}(X) = m$, $S_n = X_1 + \ldots + X_n$, THEN

$\boxed{\frac{S_n}{n} \Rightarrow m}$ PROOF: IN A FEW WEEKS.

EX: IF X_n IS UNIFORMLY DISTRIBUTED ON $\{1, 2, \ldots, n\}$, I.E. $\mathbb{P}(X_n = k) = \frac{1}{n}$, $1 \leq k \leq n$

THEN $\boxed{\frac{X_n}{n} \Rightarrow \text{UNI}[0,1]}$

PROOF: $F_n(x) = \mathbb{P}\left(\frac{X_n}{n} \leq x\right) = \begin{cases} 0 & \text{IF } x \leq 0 \\ \frac{\lfloor n \cdot x \rfloor}{n} & \text{IF } 0 \leq x \leq 1 \\ 1 & \text{IF } x \geq 1 \end{cases}$



← GRAPH OF $F_6(x)$

$\forall x \in \mathbb{R}$ $\lim\limits_{x \to \infty} F_n(x) = F(x) = \begin{cases} 0 & \text{IF } x \leq 0 \\ x & \text{IF } 0 \leq x \leq 1 \\ 1 & \text{IF } x \geq 1 \end{cases}$

COUNTER-EXAMPLE:

IF WE CONSIDER X_n FROM THE PREVIOUS EX., THEN $G_n(x) := \mathbb{P}(X_n \leq x) = \begin{cases} 0 & \text{IF } x \leq 0 \\ \frac{\lfloor x \rfloor}{n} & \text{IF } 0 \leq x \leq n \\ 1 & \text{IF } x \geq n \end{cases}$

THUS $\forall x \in \mathbb{R}$ $\lim\limits_{n \to \infty} G_n(x) = 0$

BUT $G(x) \equiv 0$ IS NOT THE C.D.F. OF A R.V.,

THUS X_n DOES NOT CONVERGE WEAKLY.

(MASS ESCAPES TO INFINITY)

EX: LET $X_1, X_2, \ldots$ I.I.D. $\mathrm{EXP}(1)$.

LET $M_n := \max\{X_1, \ldots, X_n\}$.

M_n GROWS TO INFINITY, BUT HOW FAST?

CLAIM: $M_n - \ln(n)$ CONVERGES IN DISTRIBUTION:

LET $F_n(x) := \mathbb{P}(M_n - \ln(n) \le x) =$

$$= \mathbb{P}(M_n \le \ln(n) + x) = \mathbb{P}\left(\bigcap_{r=1}^{n} \{X_r \le \ln(n) + x\}\right)$$

$$= \prod_{r=1}^{n} \mathbb{P}(X_r \le \ln(n) + x) = \left(1 - \exp(-(\ln(n) + x))\right)^n$$

$$= \left(1 - \frac{e^{-x}}{n}\right)^n, \text{ THUS } \boxed{\lim_{n \to \infty} F_n(x) = \exp(-e^{-x})}$$

$\boxed{F(x) = \exp(-e^{-x})}$ IS INDEED A C.D.F., IT IS THE C.D.F. OF THE STANDARD GUMBEL DISTRIBUTION.

EXTREME VALUE THEORY (BMETE95MM16)

$\hookrightarrow$ THEORY OF WEAK LIMITS OF THE MAXIMUM OF RANDOM VARIABLES. USED IN ACTUARIAL SCIENCE (INSURANCE MATH).

FUN FACT: IF Y_1, Y_2 I.I.D. STANDARD GUMBEL

A

THEN $\boxed{\max\{Y_1, Y_2\} \sim Y + \ln(2)}$ WHERE Y IS

INDEED: $\boxed{F^2(x) = F(x - \ln(2))}$ ✓

ALSO: IF $n \gg 1$ THEN

$$\underbrace{\max\{X_1, \dots, X_{2n}\}}_{\approx Y + \ln(2n)} = \underbrace{\max\{X_1, \dots, X_n\}}_{\approx Y_1 + \ln(n)} \vee \underbrace{\max\{X_{n+1}, \dots, X_{2n}\}}_{\approx Y_2 + \ln(n)}$$

$$\ln(2n) = \ln(2) + \ln(n)$$

CLAIM: IF $X_1, X_2, \dots$ ARE $\mathbb{Z}$-VALUED R.V.'S

THEN $\boxed{X_n \Rightarrow X}$ $\Longleftrightarrow$ $\boxed{\mathbb{P}(X_n = k) \xrightarrow[n\to\infty]{} \mathbb{P}(X = k) \quad k \in \mathbb{Z}}$

(A) (B)

PROOF: (A) $\Rightarrow$ (B): $F_n(k - \frac{1}{2}) \to F(k - \frac{1}{2})$ AS $n \to \infty$ BY (A)

$$\mathbb{P}(X_n = k) = F_n((k+1) - \tfrac{1}{2}) - F_n(k - \tfrac{1}{2})$$

$$\mathbb{P}(X = k) = F((k+1) - \tfrac{1}{2}) - F(k - \tfrac{1}{2})$$

THUS (B) HOLDS ✓

(B) $\Rightarrow$ (A): NEXT PAGE

Ⓑ $\Rightarrow$ Ⓐ: $\quad \mathbb{P}(X_n \leq x) = \sum_{k \leq x} \mathbb{P}(X_n = k)$

$$\liminf_{n\to\infty} \mathbb{P}(X_n \leq x) = \liminf_{n\to\infty} \sum_{k \leq x} \mathbb{P}(X_n = k) \overset{\text{FATOU}}{\geq}$$

$$\sum_{k \leq x} \mathbb{P}(X = k) = \mathbb{P}(X \leq x) \text{, THUS}$$

$$\boxed{\liminf_{n\to\infty} \mathbb{P}(X_n \leq x) \geq \mathbb{P}(X \leq x)} \leftarrow ☺$$

SIMILARLY:

$$\liminf_{n\to\infty} \underbrace{\mathbb{P}(X_n > x)}_{1-\mathbb{P}(X_n \leq x)} \geq \underbrace{\mathbb{P}(X > x)}_{1-\mathbb{P}(X \leq x)} \text{, THUS}$$

$$\underbrace{\liminf_{n\to\infty} \left(-\mathbb{P}(X_n \leq x)\right)}_{-\limsup_{n\to\infty} \mathbb{P}(X_n \leq x)} \geq -\mathbb{P}(X \leq x) \text{, THUS}$$

$$\boxed{\limsup_{n\to\infty} \mathbb{P}(X_n \leq x) \leq \mathbb{P}(X \leq x)} \leftarrow ☹$$

PUTTING ☺ AND ☹ TOGETHER:

$$\boxed{\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x)}$$

THUS Ⓐ HOLDS!