PROOF OF HOEFFDING USING LEMMA:

$$\mathbb{E}\left(e^{\lambda \cdot S_n}\right) = \prod_{k=1}^{n} \mathbb{E}\left(e^{\lambda \cdot X'_k}\right) \leq \prod_{i=1}^{n} \exp\left(\frac{1}{8} \cdot \lambda^2 \cdot (b_i - a_i)^2\right) =$$

$$= e^{\frac{1}{2}\sigma^2 \cdot \lambda^2}, \text{ WHERE } \sigma^2 = \frac{1}{4} \cdot \left((b_1-a_1)^2 + \ldots + (b_n - a_n)^2\right)$$

$$\mathbb{P}(S_n \geq t) = \mathbb{E}\left(e^{\lambda \cdot S_n} \geq e^{\lambda t}\right) \leq \frac{\mathbb{E}\left(e^{\lambda S_n}\right)}{e^{\lambda t}} \leq e^{\frac{1}{2}\sigma^2 \lambda^2 - \lambda t}$$

NOW $\min_{\lambda \geq 0} \left\{ \frac{1}{2}\sigma^2 \cdot \lambda^2 - \lambda t \right\} = -\frac{1}{2}\frac{t^2}{\sigma^2}$ ✓

SEE PAGE 14

EX: TOWN WITH 1000 HOUSEHOLDS, ONE GARBAGE CAN PER HOUSEHOLD. CAPACITY OF A GARBAGE CAN: 30 kg. THE AVERAGE WEEKLY GARBAGE OUTPUT MAY DIFFER FROM HOUSEHOLD TO HOUSEHOLD. AVERAGE WEEKLY OUTPUT OF TOWN IS 10^4 kg. CAPACITY OF ONE TRUCK: 10^3 kg

HOW MANY TRUCKS DO WE NEED IF WE WANT ALL GARBAGE TAKEN AWAY WITH 99% CHANCE?
(GARBAGE TRUCK COMES ONCE PER WEEK)

SOLUTION: $a_i = 0$, $b_i = 30$

$$\mathbb{P}(S_{1000} \geq 10^4 + t) \leq \exp\left(-\frac{2t^2}{1000 \cdot 30^2}\right) = 0.01 \implies$$

$$\implies t = \sqrt{\frac{1}{2}\ln(100) \cdot 9 \cdot 10^5} \approx 1440$$ WE NEED 12 TRUCKS.

THM: (BERNSTEIN'S INEQUALITY, 1937)

$X_1, X_2, \ldots, X_n$ INDEPENDENT

$S_n = X_1 + \ldots + X_n$

$D_n^2 := Var(S_n) = Var(X_1) + \ldots + Var(X_n)$

IF $\mathbb{P}(|X_r - \mathbb{E}(X_r)| \leq K) = 1$, $1 \leq r \leq n$, THEN

$$\mathbb{P}\left(S_n - \mathbb{E}(S_n) \geq t \cdot D_n\right) \leq \underbrace{\exp\left(-\frac{t^2}{2 \cdot \left(1 + \frac{t \cdot K}{2 D_n}\right)^2}\right)}_{\star}$$

FOR ANY $\boxed{0 \leq t \leq \frac{D_n}{K}}$

COROLLARY: (BY APPLYING THM TO $-S_n$):

$\mathbb{P}(S_n - \mathbb{E}(S_n) \leq -t \cdot D_n) \leq \star$, THUS

$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq t \cdot D_n) \leq 2 \cdot \star$

NOTE: BERNSTEIN IS QUITE SHARP IF $X_1, X_2, \ldots$ ARE I.I.D., $n \to \infty$, BUT t IS FIXED:

$$\limsup_{n \to \infty} \mathbb{P}\left(\frac{S_n - \mathbb{E}(S_n)}{D_n} \geq t\right) \leq \exp\left(-\frac{t^2}{2}\right)$$

COMPARE THIS TO ...

COMPARE THIS TO CENTRAL LIMIT THM:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - \mathbb{E}(S_n)}{D_n} \geq t \right) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \leq$$

$$\leq \int_t^\infty e^{-x^2/2}\, dx \underset{\boxed{t \geq 1}}{\leq} \int_t^\infty x \cdot e^{-x^2/2}\, dx = e^{-t^2/2}$$

PROOF OF BERNSTEIN: W.L.O.G.: $\mathbb{E}(X_i) = 0$

LEMMA: IF $\mathbb{P}(|X| \leq K) = 1$, $Var(X) = \sigma^2$, $\mathbb{E}(X) = 0$, THEN

$$\boxed{\hat{I}(\lambda) \leq \frac{\lambda^2 \cdot \sigma^2}{2} \cdot \left(1 + \frac{\lambda \cdot K \cdot e^{\lambda K}}{3} \right)}$$

PROOF: $Z(\lambda) = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \mathbb{E}(X^r) =$

$$= 1 + 0 + \frac{\lambda^2}{2} \cdot \sigma^2 + \sum_{r=3}^{\infty} \frac{\lambda^r}{r!} \mathbb{E}(X^r) = ☆$$

$\boxed{k \geq 3}$: $\mathbb{E}(X^r) \leq K^{k-2} \cdot \mathbb{E}(X^2) = K^{r-2} \cdot \sigma^2$, THUS

$$\frac{\lambda^r}{k!} \mathbb{E}(X^r) \leq \frac{\lambda^2 \cdot \sigma^2}{2} \cdot \frac{\lambda \cdot K}{3} \cdot \frac{(\lambda \cdot K)^{r-3}}{\underbrace{4 \cdot 5 \cdot \ldots \cdot k}_{\geq (r-3)!}}$$

$$☆ \leq 1 + \frac{\lambda^2}{2} \cdot \sigma^2 \cdot \left(1 + \frac{\lambda K}{3} \sum_{r=3}^{\infty} \frac{(\lambda \cdot K)^{r-3}}{(r-3)!} \right) \quad \text{=====} \quad ☺$$

$$☺ = 1 + \frac{\lambda^2}{2} \cdot \sigma^2 \cdot \left(1 + \frac{\lambda K}{3} e^{\lambda K}\right) \leq$$

$\boxed{\text{SINCE } 1 + x \leq e^x}$

$$\leq \exp\left(\frac{\lambda^2}{2} \cdot \sigma^2 \cdot \left(1 + \frac{\lambda K}{3} e^{\lambda K}\right)\right) \checkmark$$

PROOF OF BERNSTEIN USING LEMMA:

$$\mathbb{E}\left(e^{\lambda \cdot S_n}\right) = \prod_{r=1}^{n} \mathbb{E}\left(e^{\lambda \cdot X_r}\right) \leq \exp\left(\frac{\lambda^2}{2} \cdot D_n^2 \cdot \left(1 + \frac{\lambda K}{3} \cdot e^{\lambda K}\right)\right)$$

$$\mathbb{E}\left(e^{\lambda \cdot S_n / D_n}\right) \overset{\boxed{\text{HW1.3(a)}}}{\leq} \exp\left(\frac{\lambda^2}{2} \cdot \left(1 + \frac{\lambda \cdot K / D_n}{3} \cdot e^{\lambda \cdot K / D_n}\right)\right)$$

MARKOV INEQ.

$$\mathbb{P}\left(S_n \geq t \cdot D_n\right) = \mathbb{P}\left(e^{\lambda \cdot S_n / D_n} > e^{\lambda t}\right) \leq$$

$$\leq \inf_{\lambda > 0} \exp\left(\frac{\lambda^2}{2} \cdot \left(1 + \frac{\lambda \cdot K / D_n}{3} \cdot e^{\lambda K / D_n}\right) - \lambda \cdot t\right)$$

$$\text{LET } \boxed{\lambda := \frac{t}{\left(1 + \frac{t \cdot K}{2 D_n}\right)^2}}$$

NOTE: $\boxed{\lambda < t}$

ASSUMED: $\boxed{t \leq \frac{D_n}{K}} \Rightarrow \boxed{\frac{t \cdot K}{D_n} \leq 1}$

THEN:

$$\frac{\lambda^2}{2} \cdot \left(1 + \frac{\lambda \cdot K}{D_n} \cdot \frac{e^{\lambda K / D_n}}{3}\right) - \lambda \cdot t =$$

$$\frac{t^2}{2 \cdot \left(1 + \frac{t \cdot K}{2 D_n}\right)^2} \cdot \frac{1 + \frac{\lambda K}{D_n} \cdot \frac{e^{\lambda K / D_n}}{3}}{\left(1 + \frac{t \cdot K}{2 D_n}\right)^2} - \frac{t^2}{\left(1 + \frac{t \cdot K}{2 D_n}\right)^2} = ★$$

$$\bigstar = \frac{t^2}{2\cdot\left(1+\frac{t\cdot K}{2D_n}\right)^2} \cdot \frac{1+\frac{\lambda K}{D_n}\cdot\frac{e^{\lambda K/D_n}}{3}}{\left(1+\frac{t\cdot K}{2D_n}\right)^2} - \frac{t^2}{\left(1+\frac{t\cdot K}{2D_n}\right)^2}$$

NOTE: $e^{\lambda K/D_n} \overset{\boxed{\lambda<t}}{\leq} e^{t\cdot K/D_n} \leq e^1 < 3$, THUS

$$\frac{1+\frac{\lambda K}{D_n}\cdot\frac{e^{\lambda K/D_n}}{3}}{\left(1+\frac{t\cdot K}{2D_n}\right)^2} \leq \frac{1+\frac{\lambda\cdot K}{D_n}}{1+\frac{t\cdot K}{D_n}+\left(\frac{t K}{2D_n}\right)^2} \overset{\boxed{\lambda<t}}{\leq} 1$$

$$\bigstar \leq \frac{t^2}{2\cdot\left(1+\frac{tK}{2D_n}\right)^2} - \frac{t^2}{\left(1+\frac{tK}{2D_n}\right)^2} = \frac{-t^2}{2\cdot\left(1+\frac{t\cdot K}{2D_n}\right)^2} \checkmark$$

SOME MEASURE THEORY:

FACT: MONOTONE CONVERGENCE THM:

IF $0 \leq X_1 \leq X_2 \leq \ldots \leq X_n \leq \ldots$

THEN $\boxed{\mathbb{E}\left(\sup_n X_n\right) = \sup_n \mathbb{E}(X_n)}$

(AND EQUALITY HOLDS EVEN IN THE CASE WHEN $\mathbb{E}\left(\sup_n X_n\right) = +\infty$)

LEMMA (FATOU): IF $X_m \geq 0 \quad \forall m \in \mathbb{N}$

THEN $\boxed{\mathbb{E}\left(\liminf_{n\to\infty} X_m\right) \leq \liminf_{n\to\infty} \mathbb{E}(X_m)}$

PROOF: $\liminf_{n\to\infty} X_m = \sup_n \inf_{m \geq n} X_m$

LET $\boxed{Y_n := \inf_{m \geq n} X_m}$, THEN

$\boxed{0 \leq Y_1 \leq Y_2 \leq \dots}$ NOTE: $\boxed{Y_n \leq X_n}$

$$\mathbb{E}\left(\liminf_{n\to\infty} X_m\right) = \mathbb{E}\left(\sup_n Y_n\right) \overset{\text{MONOTONE CONV.}}{=}$$

$$= \sup_n \mathbb{E}(Y_n) = \lim_{n\to\infty} \mathbb{E}(Y_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n)$$

THM (DOMINATED CONVERGENCE):

IF $\lim_{n\to\infty} X_n$ EXISTS AND $\forall n \in \mathbb{N}$

$\boxed{|X_n| \leq Y}$, WHERE $\boxed{\mathbb{E}(Y) < +\infty}$ THEN

$\boxed{\mathbb{E}\left(\lim_{n\to\infty} X_n\right) = \lim_{n\to\infty} \mathbb{E}(X_n)}$

PROOF: APPLY FATOU TO $Y + X_n$ AND $Y - X_n$

$$\boxed{X := \lim_{n\to\infty} X_n} \qquad \boxed{Z_n := Y + X_n} \qquad \text{NOTE: } \boxed{Z_n \geq 0}$$

$$\mathbb{E}(Y) + \mathbb{E}(X) = \mathbb{E}(Y + X) = \mathbb{E}\left(\lim_{n\to\infty} Z_n\right) =$$

$$\mathbb{E}\left(\liminf_{n\to\infty} Z_n\right) \overset{\text{FATOU}}{\leq} \liminf_{n\to\infty} \mathbb{E}(Z_n) =$$

$$\liminf_{n\to\infty} \left(\mathbb{E}(Y) + \mathbb{E}(X_n)\right) = \mathbb{E}(Y) + \liminf_{n\to\infty} \mathbb{E}(X_n)$$

THUS $\boxed{\mathbb{E}(X) \leq \liminf_{n\to\infty} \mathbb{E}(X_n)} \Leftarrow$ ☺

NOW LET $\boxed{\tilde{Z}_n := Y - X_n}$ NOTE: $\boxed{\tilde{Z}_n \geq 0}$

$$\mathbb{E}(Y) - \mathbb{E}(X) = \mathbb{E}\left(\liminf_{n\to\infty} \tilde{Z}_n\right) \overset{\text{FATOU}}{\leq}$$

$$\leq \liminf_{n\to\infty} \mathbb{E}(\tilde{Z}_n) = \liminf_{n\to\infty} \left(\mathbb{E}(Y) - \mathbb{E}(X_n)\right)$$

$$= \mathbb{E}(Y) + \liminf_{n\to\infty} \left(-\mathbb{E}(X_n)\right) = \mathbb{E}(Y) - \limsup_{n\to\infty} \mathbb{E}(X_n)$$

$$\boxed{-\mathbb{E}(X) \leq -\limsup_{n\to\infty} \mathbb{E}(X_n)} \Rightarrow \boxed{\mathbb{E}(X) \geq \limsup_{n\to\infty} \mathbb{E}(X_n)}$$

☹

NOW ☹ AND ☺ TOGETHER GIVE:

$$\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n) \quad \checkmark$$