THUS:

$$\hat{I}''(\lambda) = \frac{z''(\lambda)}{z(\lambda)} - \left(\frac{z'(\lambda)}{z(\lambda)}\right)^2 = \mathbb{E}\left(\left(X^{(\lambda)}\right)^2\right) - \left(\mathbb{E}\left(X^{(\lambda)}\right)\right)^2$$

$$\boxed{\hat{I}''(\lambda) = \mathrm{Var}\left(X^{(\lambda)}\right) > 0}$$

INDEED: $\hat{I}$ STRICTLY CONVEX!

THUS BY PAGE 12-13:

IF WE DEFINE $I(x) = \sup_{\lambda} \{\lambda \cdot x - \hat{I}(\lambda)\}$

THEN INDEED $\hat{I}(\lambda) = \sup_{x} \{\lambda \cdot x - I(x)\}$,

MOREOVER

$\lambda \longmapsto \hat{I}'(\lambda)$ IS STRICTLY INCREASING, CONTINUOUS

$x \longmapsto I'(x)$ IS — " — — " — — " —

$\boxed{I' = (\hat{I}')^{-1}}$ AND IF WE DEFINE $\boxed{\lambda^*(x) := (\hat{I}')^{-1}(x)}$

THEN $\mathbb{E}\left(X^{(\lambda^*(x))}\right) = \hat{I}'(\lambda^*(x)) = x$ AND

$$\boxed{I(x) = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x))}, \quad \boxed{I''(x) = \frac{1}{\mathrm{Var}\left(X^{(\lambda^*(x))}\right)}}$$

NOTE: $\hat{I}(0) = 0$, $\hat{I}'(0) = \mathbb{E}(X) = m$, THUS

$\lambda^*(m) = 0$, THUS $I(m) = X \cdot 0 - \hat{I}(0) = 0$

(($I'(m)$

NOTE: IF $\boxed{x \geq m}$ THEN $\lambda^*(x) \geq \lambda^*(m) = 0$

THUS $I(x) = \max\limits_{\lambda \geq 0} \{\lambda \cdot x - \hat{I}(\lambda)\}$, THUS

BY PAGE 8: $\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \leq e^{-n \cdot I(x)}$

LEMMA: IF X AND Y ARE $\underbrace{\text{INDEPENDENT}}$

AND $X^{(\lambda)}$ AND $Y^{(\lambda)}$ ARE —"—

THEN $\boxed{(X+Y)^{(\lambda)} \sim X^{(\lambda)} + Y^{(\lambda)}}$

PROOF: WE ONLY SHOW IT IN THE CASE WHEN BOTH X AND Y ARE ABS. CONTINUOUS:

f IS THE DENSITY F'N OF X

g IS —"— —"— —"— OF Y

$f * g$ IS —"— —"— —"— OF $X + Y$

WANT: $\boxed{(f*g)^{(\lambda)} = f^{(\lambda)} * g^{(\lambda)}}$

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy \quad \text{(CONVOLUTION)}$$

NOTE: $Z_{X+Y}(\lambda) = \boxed{\text{HW } 1.3(b)} = Z_X(\lambda) \cdot Z_Y(\lambda)$

$$(f*g)^{(\lambda)}(x) = \frac{e^{\lambda x}}{Z_{X+Y}(\lambda)} \cdot \int_{-\infty}^{\infty} f(y) \cdot g(x-y)\, dy =$$

$$= \int_{-\infty}^{\infty} \left(\frac{e^{\lambda y}}{Z_X(\lambda)} \cdot f(y)\right) \cdot \left(\frac{e^{\lambda \cdot (x-y)}}{Z_Y(\lambda)} \cdot g(x-y)\right) dy = \left(f^{(\lambda)} * g^{(\lambda)}\right)(x)$$

✓

<u>REMINDER</u>: $X_1, X_2, \dots$ I.I.D., $S_n = X_1 + \dots + X_n$

$$Z(\lambda) = \mathbb{E}\left(e^{\lambda X_i}\right), \quad \hat{I}(\lambda) = \ln\left(Z(\lambda)\right)$$

B

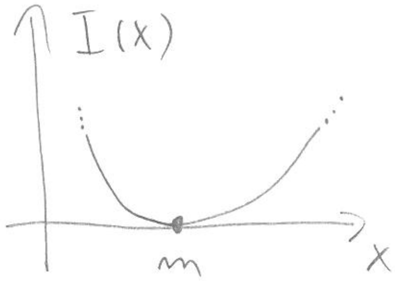
$$I(x) = \max_{\lambda} \left\{ x\lambda - \hat{I}(\lambda) \right\} = x \cdot \lambda^*(x) - \hat{I}(\lambda^*(x)),$$

$$\lambda^*(x) = \left(\hat{I}'\right)^{-1}(x), \quad \mathbb{E}\left(X_i^{(\lambda^*(x))}\right) = x, \quad \text{WHERE}$$

$$\mathbb{P}\left(X_i^{(\lambda)} \in A\right) = \mathbb{E}\left(\frac{e^{\lambda X_i}}{Z(\lambda)} \cdot \mathbb{1}[X_i \in A]\right), \quad A \subseteq \mathbb{R}$$

<u>KNOW</u>: $x \mapsto I(x)$ IS STRICTLY CONVEX

$$m = \mathbb{E}(X_i) \qquad I'(m) = 0, \quad I(m) = 0, \quad I''(x) > 0$$



THUS $I(x) > 0$ IF $x \neq m$

<u>THM</u> (H. CRAMÉR): $a < b \in \mathbb{R}$:

(U) $$\limsup_{n \to \infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \in [a,b]\right)\right) \leq - \inf_{a \leq x \leq b} I(x)$$

(L) $$\liminf_{n \to \infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \in (a,b)\right)\right) \geq - \inf_{a < x < b} I(x)$$

REMARK: IF $X_i \sim BER(p)$ THEN

$$I(x) = \begin{cases} x \cdot \ln\left(\frac{x}{p}\right) + (1-x)\cdot \ln\left(\frac{1-x}{1-p}\right) & \text{IF } 0 \leq x \leq 1 \\ +\infty & \text{IF } x \notin [0,1] \end{cases}$$

$$\inf_{1 \leq x \leq 2} I(x) = I(1) = \ln(p) \iff \mathbb{P}\left(\frac{S_n}{n} \geq 1\right) = p^n = e^{-n \cdot I(1)}$$

$$\inf_{1 < x < 2} I(x) = +\infty \iff \mathbb{P}\left(\frac{S_n}{n} > 1\right) = 0 = e^{-n \cdot \infty}$$

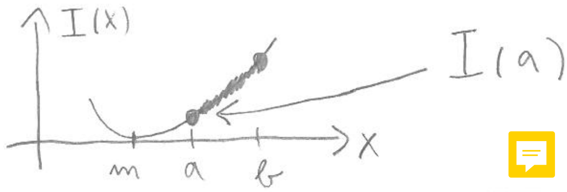
PROOF: (U): IF $m \in [a,b]$ THEN $\inf_{a \leq x \leq b} I(x) = I(m) = 0$

THUS $\mathbb{P}\left(\frac{S_n}{n} \in [a,b]\right) \leq 1 = e^{-n \cdot I(m)}$ ✓

WITHOUT LOSS OF GENERALITY, ASSUME $\boxed{m < a}$:

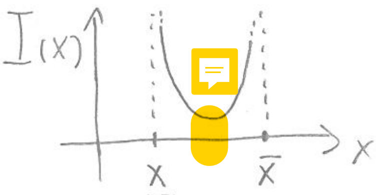
$$\mathbb{P}\left(\frac{S_n}{n} \in [a,b]\right) \leq \mathbb{P}\left(\frac{S_n}{n} \geq a\right) \leq e^{-n \cdot I(a)} \quad \text{(PAGE 20)}$$

AND $I(a) = \inf_{a \leq x \leq b} I(x)$ ✓



(L): IF $\underline{x} = \inf\{x : I(x) < +\infty\}$

$\bar{x} = \sup\{x : I(x) < +\infty\}$



THEN $x \in (\underline{x}, \bar{x}) \implies \boxed{I(x) < +\infty}$ $\boxed{|I'(x)| < +\infty}$

(SINCE I IS CONVEX, SMOOTH)

PAGE 22

IF $(a,b) \cap (\underline{x}, \overline{x}) = \emptyset$ THEN $\inf\limits_{a<x<b} I(x) = +\infty$

THUS (L) TRIVIALLY FOLLOWS.

WE WILL SHOW THAT IF $x \in (a,b) \cap (\underline{x}, \overline{x})$,

THEN $\boxed{\liminf\limits_{n\to\infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \in (a,b)\right)\right) \geq -I(x)}$

(☆) →

AS SOON AS WE SHOW (☆), THE PROOF OF (L) FOLLOWS.

<u>NOTE</u>: IF ε IS SMALL THEN $[x-\varepsilon, x+\varepsilon] \subseteq (a,b)$

LET $\lambda^* = \lambda^*(x) := I'(x)$, THEN $\mathbb{E}\left(X_i^{(\lambda^*)}\right) = x$

ALSO NOTE: $\boxed{X_1^{(\lambda^*)} + \ldots + X_n^{(\lambda^*)} \sim S_n^{(\lambda^*)}}$ (SEE PAGE 20)

WEAK LAW OF LARGE NUMBERS

THUS $\lim\limits_{n\to\infty} \mathbb{P}\left(\frac{S_n^{(\lambda^*)}}{n} \in [x-\varepsilon, x+\varepsilon]\right) \underset{A}{=} 1$, BUT

$\mathbb{P}\left(S_n^{(\lambda^*)} \in [n\cdot(x-\varepsilon), n\cdot(x+\varepsilon)]\right) =$ (SEE PAGE 17)

$\mathbb{E}\left(\frac{e^{\lambda^*\cdot S_n}}{Z(\lambda^*)^n} \cdot \mathbb{1}\left[S_n \in [n\cdot(x-\varepsilon), n\cdot(x+\varepsilon)]\right]\right) \leq$

$\boxed{\begin{array}{l}\lambda^* > 0 \\ \text{SINCE} \\ m < a < x\end{array}}$

$\frac{e^{\lambda^*\cdot(x+\varepsilon)\cdot n}}{(Z(\lambda^*))^n} \cdot \underbrace{\mathbb{P}\left(\frac{S_n}{n} \in [x-\varepsilon, x+\varepsilon]\right)}_{\leq \mathbb{P}\left(\frac{S_n}{n} \in (a,b)\right)}$

THUS: $\forall \varepsilon > 0$ (SMALL):

$$0 = \lim_{n \to \infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n^{(\lambda^*)}}{n} \in [x-\varepsilon, x+\varepsilon] \right) \right) \leq$$

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{e^{\lambda^* \cdot (x+\varepsilon)\cdot n}}{(Z(\lambda^*))^n} \right)}_{= \lambda^* \cdot x + \lambda^* \cdot \varepsilon - \hat{I}(\lambda^*) = I(x) + \lambda^* \cdot \varepsilon} + \liminf_{n \to \infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \in (a,b) \right) \right)$$

SEE PAGE 21

SINCE IT HOLDS FOR ANY $\varepsilon > 0$, WE ARE DONE WITH THE PROOF OF ★. WE USED MEASURE CHANGE.

REMARK: IF $x \in (\underline{x}, \bar{x})$ THEN

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln\left(\mathbb{P}\left(\frac{S_n}{n} \in [x-\varepsilon, x+\varepsilon] \right) \right) = -I(x)$$

HEURISTICALLY: $\boxed{\mathbb{P}\left(\frac{S_n}{n} \approx x \right) \approx e^{-n \cdot I(x)}}$

$I(x)$ IS THE „COST" OF $\frac{S_n}{n} \approx x$

CRAMÉR: IF THE UNLIKELY EVENT $\frac{S_n}{n} \in [a,b]$ OCCURS, THEN IT WILL OCCUR USING THE LEAST COSTLY STRATEGY.

THE BEST STRATEGY TO ACHIEVE

$\frac{S_n}{n} \approx x$ IS TO TILT WITH $\lambda^*(x)$:

CLAIM: IF X_i IS INTEGER-VALUED

WITH $\mathbb{P}(X_i = r) = p_r$, $\boxed{x > m}$, THEN

$$\lim_{n\to\infty} \mathbb{P}(X_1 = r \mid S_n \geq n\cdot x) = p_r^{(\lambda^*(x))} = \frac{e^{\lambda^*(x)\cdot r}\cdot p_r}{Z(\lambda^*(x))}$$

PROOF (NON-RIGOROUS): FIRST NOTE THAT IF $\lambda^* = \lambda^*(x)$

$$\mathbb{P}(X_1^{(\lambda^*)} = r \mid S_n^{(\lambda^*)} = m) \underset{\uparrow}{=} \mathbb{P}(X_1 = r \mid S_n = m)$$

$\boxed{\text{HW } 2.3}$

THUS $\mathbb{P}(X_1 = r \mid S_n \geq n\cdot x) \approx$ CRAMÉR

$$\mathbb{P}(X_1 = r \mid S_n \approx n\cdot x) \approx$$

$$\mathbb{P}(X_1^{(\lambda^*)} = r \mid S_n^{(\lambda^*)} \approx n\cdot x) =$$

$$= \frac{p_r^{(\lambda^*)} \cdot \mathbb{P}(X_2^{(\lambda^*)} + \dots + X_n^{(\lambda^*)} \approx n\cdot x - r)}{\mathbb{P}(X_1^{(\lambda^*)} + \dots + X_n^{(\lambda^*)} \approx n\cdot x)} \approx$$

WEAK LAW OF LARGE NUMBERS

$$\approx \frac{p_r^{(\lambda^*)} \cdot 1}{1} = p_r^{(\lambda^*)}$$