DEF: LET $F(x) = \mathbb{P}(X \leq x)$, $X \sim F$

WE SAY THAT THE DISTRIBUTON OF X IS STABLE IF

$\forall r \in \mathbb{N}$ $\exists$ $\alpha_r > 0$, $\beta_r \in \mathbb{R}$ SUCH THAT IF $X_1, \dots, X_r$ I.I.D., $X_i \sim F$

THEN
$$\boxed{\frac{X_1 + \dots + X_r}{\alpha_r} - \beta_r \sim F}$$
A

REMARK: IF X IS STABLE THEN $a \cdot X + b$ IS STABLE $\forall a, b \in \mathbb{R}$

EXAMPLES: IF $X \sim \mathcal{N}(0,1)$ THEN

$$\frac{X_1 + \dots + X_r}{\sqrt{r}} \sim \mathcal{N}(0,1)$$
B

THUS $\mathcal{N}(0,1)$ IS STABLE!

THUS $\mathcal{N}(\mu, \sigma)$ IS ALSO STABLE.

$$\boxed{\varphi(t) = e^{-t^2/2}}$$
C

IF $X_1 \sim CAU(1)$ (STANDARD CAUCHY)

THEN $\frac{X_1 + \dots + X_n}{n} \sim CAU(1)$ THUS CAUCHY IS STABLE!

A

$\boxed{\varphi(t) = e^{-|t|}}$ B

SEE PAGE 105-106-107

IF $X_1 \sim$ LÉVY (SEE PAGE 61)

THEN $\boxed{\frac{X_1 + \dots + X_n}{n^2} \sim \text{LÉVY}}$ C THUS LÉVY IS STABLE!

PROOF: HW 7.2 (d) : $\boxed{\varphi(t) = e^{-\sqrt{-2it}}}$ D

$$\mathbb{E}\left[\exp\left(i \cdot t \cdot \left(\frac{X_1 + \dots + X_n}{n^2}\right)\right)\right] \overset{E}{=} \varphi\left(\frac{t}{n^2}\right)^n \overset{F}{=}$$

$$= e^{n \cdot \left(-\sqrt{-2it/n^2}\right)} \overset{G}{=} e^{-\sqrt{-2it}} = \varphi(t) \checkmark$$

THM: IF $X_1, X_2, \dots$ I.I.D., $S_n = X_1 + \dots + X_n$

IF THERE EXIST $a_n \in \mathbb{R}_+$, $b_n \in \mathbb{R}$ SUCH THAT

$\boxed{\frac{S_n - b_n}{a_n} \Rightarrow Z_1}$ H THEN Z_1 IS STABLE.

PAGE 141

PROOF:

FIX r AND LET

$$S_n^{(j)} := \sum_{n\cdot(j-1) < i \le n\cdot j} X_i \tag{A}$$

$j = 1, \dots, r$

THEN $S_n^{(1)}, S_n^{(2)}, \dots, S_n^{(r)}$ ARE I.I.D.

LET

$$Z_n^{(j)} := \frac{S_n^{(j)} - b_n}{a_n} \tag{B}$$

$$Z_{r\cdot n} := \frac{S_{r\cdot n} - b_{r\cdot n}}{a_{r\cdot n}} \tag{C}$$

THUS

$$\underbrace{S_n^{(1)} + \dots + S_n^{(r)}}_{a_n\cdot\left(Z_n^{(1)} + \dots + Z_n^{(r)}\right) + r\cdot b_n} = \underbrace{S_{r\cdot n}}_{a_{r\cdot n}\cdot Z_{r\cdot n} + b_{r\cdot n}} \tag{D}$$

$$a_n\cdot\left(Z_n^{(1)} + \dots + Z_n^{(r)}\right) + r\cdot b_n = a_{r\cdot n}\cdot Z_{r\cdot n} + b_{r\cdot n} \tag{E}$$

THUS

$$\frac{a_n}{a_{r\cdot n}}\cdot\left(Z_n^{(1)} + \dots + Z_n^{(k)}\right) - \left(\frac{b_{r\cdot n} - k\cdot b_n}{a_{r\cdot n}}\right) = Z_{r\cdot n} \tag{F}$$

NOW LET $n \to \infty$: $\quad Z_n^{(1)} \Downarrow Z^{(1)}, \ \dots, \ Z_n^{(k)} \Downarrow Z^{(r)}, \quad Z_{r\cdot n} \Downarrow Z_1$

$$Z^{(1)} + \dots + Z^{(r)} \qquad\qquad Z_1$$

THUS IF WE CAN PROVE THAT

$$\lim_{n\to\infty} \frac{a_n}{a_{r\cdot n}} =: \frac{1}{d_r} \tag{G}$$

$$\lim_{n\to\infty} \frac{b_{r\cdot n} - r\cdot b_n}{a_{r\cdot n}} =: \beta_r \tag{H}$$

THEN

$$\frac{Z^{(1)} + \dots + Z^{(r)}}{d_r} - \beta_r \sim Z_1 \tag{I}$$

THUS Z IS STABLE.

PAGE 142

THUS WE ONLY NEED:

LEMMA: LET $\boxed{W_n \Rightarrow W}$ A AND

LET $\alpha_n \in \mathbb{R}_+$, $\beta_n \in \mathbb{R}$, $\boxed{W_n^* = \alpha_n \cdot W_n + \beta_n}$ B

IF $\boxed{W_n^* \Rightarrow W^*}$ C AND IF BOTH W AND W^* ARE NON-DEGENERATE R.V.'S, THEN THE LIMITS $\boxed{\lim_{n\to\infty} \alpha_n =: \alpha}$ D, $\boxed{\lim_{n\to\infty} \beta_n =: \beta}$ E EXIST.

PROOF:

NOTE: $\boxed{\liminf_{n\to\infty} \alpha_n > 0}$ F

INDEED: IF (n') IS A SUBSEQUENCE FOR WHICH $\lim_{n'\to\infty} \alpha_{n'} = 0$ G THEN $\alpha_{n'} \cdot W_{n'} \Rightarrow 0$ H

THUS $W^* = \lim_{n'\to\infty} \beta_{n'}$ I, THUS W^* IS DEGENERATE ↯

SIMILARLY: $\limsup_{n\to\infty} \alpha_n < +\infty$ J **INDEED:**

IF $\lim_{n'\to\infty} \alpha_{n'} = +\infty$ K THEN $\frac{1}{\alpha_{n'}} W_n^* \Rightarrow 0$ L, THUS

$W = \lim_{n'\to\infty} \frac{-\beta_{n'}}{\alpha_{n'}}$ M, THUS W IS DEGENERATE ↯

THUS $\exists K \in (0, +\infty)$ SUCH THAT

$$\boxed{\forall m \in \mathbb{N} : 1/K \underset{A}{\le} \alpha_m \underset{B}{\le} K} \to (\alpha_m \cdot W_m) \text{ IS}$$

NOW (W_m) IS TIGHT, THUS ALSO TIGHT, THUS $(|\beta_m|)$ MUST BE BOUNDED, OTHERWISE WE WOULD HAVE $W^*_{m'} = \alpha_{m'} \cdot W_{m'} + \beta_{m'} \underset{C}{\Longrightarrow} \pm\infty$ FOR SOME SUBSEQUENCE (m'), WHICH CONTRADICTS THE FACT THAT W^* IS NON-DEG.

THUS $\exists K \in (0, +\infty)$ SUCH THAT

$$\boxed{\forall m \in \mathbb{N} : 1/K \underset{C}{\le} \alpha_m \underset{D}{\le} K, \quad -K \underset{E}{\le} \beta_m \underset{F}{\le} K}$$

NOW IF (α_m, β_m) DOES NOT CONVERGE THEN THERE EXIST TWO SUBSEQUENTIAL LIMITS (α, β) AND $(\tilde{\alpha}, \tilde{\beta})$.

THUS BY SLUTSKY:

$$\alpha \cdot W + \beta \underset{G}{\sim} W^* \underset{H}{\sim} \tilde{\alpha} \cdot W + \tilde{\beta}, \quad \text{THUS}$$

$\alpha, \tilde{\alpha} \in \mathbb{R}_+$

$\beta, \tilde{\beta} \in \mathbb{R}$

$$\star \to \boxed{W \underset{I}{\sim} \frac{\alpha}{\tilde{\alpha}} \cdot W + \frac{\beta - \tilde{\beta}}{\tilde{\alpha}}}$$

W.L.O.G ASSUME $\tilde{\alpha} \geq \alpha$

IF $\tilde{\alpha} > \alpha$ THEN $\left(\frac{\alpha}{\tilde{\alpha}}\right)^{n} \xrightarrow[n \to \infty]{} 0$, THUS

ITERATING ★ WE WOULD OBTAIN THAT

W IS DEGENERATE.

THUS $\tilde{\alpha} = \alpha$ AND $\boxed{W \sim W + (\beta - \tilde{\beta})/\alpha}$

THUS WE MUST HAVE $\beta = \tilde{\beta}$ OTHERWISE

$W = \pm \infty$ ↯

<u>THM</u>: THESE TWO CONDITIONS ARE EQUIV:

1. X IS STABLE AND $\boxed{-X_1 \sim X_1}$ (SYMMETRIC)

2. $\exists\, c > 0$ AND $\alpha \in (0, 2]$ SUCH THAT

$$\varphi(t) = \mathbb{E}\left(e^{itX_1}\right) = \exp\left(-c \cdot |t|^{\alpha}\right)$$

<u>PROOF</u>: LATER.

<u>REMARKS</u>: c CAN BE CHANGED BY SCALING.

α IS THE <u>INDEX</u> OF X.

REMARK: INDEED STABLE: $\dfrac{X_1 + \dots + X_k}{k^{1/\alpha}} \underset{A}{\sim} X_1$

PROOF: $\varphi\left(\dfrac{t}{k^{1/\alpha}}\right)^k \underset{B}{=} \exp\left(-c \cdot k \cdot \left|\dfrac{t}{k^{1/\alpha}}\right|^\alpha\right) \underset{C}{=} \varphi(t)$ ✓

WE WILL PROVE THAT $e^{-c \cdot |t|^\alpha}$ IS INDEED THE CHAR. FUNCTION OF A R.V. AND IF $\varphi(t) = \mathbb{E}(e^{itX})$ AND X_1 IS SYMMETRIC STABLE, THEN $\varphi(t) = e^{-c \cdot |t|^\alpha}$.

NOTE: $\underset{D}{\boxed{\alpha = 2}}$ NORMAL, $\underset{E}{\boxed{\alpha = 1}}$ CAUCHY.

PROPOSITION: IF $\alpha \in (0, 1]$ THEN $\varphi(t) = \underset{F}{e}^{-c \cdot |t|^\alpha}$ IS A CHARACTERISTIC FUNCTION.

PROOF: $\varphi(0) \underset{G}{=} 1$, $\varphi(-t) \underset{H}{=} \varphi(t)$,

$\lim\limits_{t \to \infty} \varphi(t) \underset{I}{=} 0$, $\varphi : (0, +\infty) \to \mathbb{R}$ IS CONVEX:

$\varphi''(t) \underset{J}{>} 0$ (HERE WE USED $0 < \alpha \leq 1$)

THUS BY $\boxed{\text{HW 9.3}}$ φ IS A CHAR. FUNCTION.

$\boxed{\text{PAGE 146}}$

REMARK: IF $\mathbb{E}(e^{itX}) \underset{A}{=} e^{-c\cdot|t|^{\alpha}}$, $\alpha \in (0,2]$

THEN THERE IS NO SIMPLE EXPLICIT FORMULA FOR THE P.D.F. OF X

(EXCEPT FOR THE $\alpha=2$ AND $\alpha=1$ CASES)

REMARK: IF $\boxed{\alpha > 2}$ THEN $\varphi(t) \underset{B}{=} e^{-c\cdot|t|^{\alpha}}$

IS NOT A CHARACTERISTIC FUNCTION.

NAIVE PROOF: $\varphi''(0) \underset{C}{=} 0$, THUS

$\mathbb{E}(X^2) \underset{D}{=} 0$ (SEE PAGE 90), BUT

THEN $\mathbb{P}(X=0) \underset{E}{=} 1$ THUS $\mathbb{E}(e^{itX}) \underset{F}{\equiv} 1$ ↯

BUT THIS NAIVE PROOF HAS A GAP:

THE THEOREM FROM PAGE 90 ONLY SAID SOMETHING ABOUT $\varphi''(0)$ IF WE ALSO ASSUMED $\mathbb{E}(X^2) \underset{G}{<} +\infty$.

WE WILL FIX THIS GAP IN $\boxed{\text{HW 10.1}}$.