THM (ERDŐS–KAC, 1940)

LET $\mathbb{P}(U_n = k) \underset{A}{=} \frac{1}{n}$, $k \in \{1, 2, \dots, n\}$

LET Z_n DENOTE THE NUMBER OF PRIME DIVISORS OF U_n. THEN

$$\boxed{\frac{Z_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \underset{B}{\Longrightarrow} \mathcal{N}(0,1)}$$

PROOF: DENOTE BY $\mathcal{P}$ THE SET OF PRIMES

CLAIM:

$$\boxed{\sum_{\substack{p \in \mathcal{P} \\ p \leq n}} \frac{1}{p} \underset{C}{=} \ln(\ln(n)) + \mathcal{O}(1)}$$

PROOF: (WE WILL ONLY PROVE THE $\geq$ PART)

$$\ln(n) \underset{D}{=} \int_1^n \frac{1}{x}\,dx \underset{E}{\leq} \sum_{k=1}^{n} \frac{1}{k} \underset{F}{\leq} \prod_{\substack{p \in \mathcal{P} \\ p \leq n}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right)$$

$$\underset{G}{=} \prod_{\substack{p \in \mathcal{P} \\ p \leq n}} \frac{1}{1 - \frac{1}{p}}, \quad \text{THUS } \ln(\ln(n)) \underset{H}{\leq} \sum_{\substack{p \in \mathcal{P} \\ p \leq n}} -\ln\left(1 - \frac{1}{p}\right) \underset{I}{=}$$

$$\underset{J}{=} \sum_{\substack{p \in \mathcal{P} \\ p \leq n}} -\left(-\frac{1}{p} + \mathcal{O}\left(\frac{1}{p^2}\right)\right) \underset{K}{=} \sum_{\substack{p \in \mathcal{P} \\ p \leq n}} \frac{1}{p} + \mathcal{O}(1) \checkmark$$

BACK TO PROOF OF ERDŐS-KAC:

LET $\boxed{Y_{n,p} \underset{A}{:=} \mathbb{1}[\,p \text{ DIVIDES } U_n\,]}$

THEN $\boxed{Z_n \underset{B}{=} \sum_{\substack{p\in\mathbb{P}\\ p\le n}} Y_{n,p}}$

$$\mathbb{E}(Y_{n,p}) \underset{C}{=} \mathbb{P}(\,p \text{ DIVIDES } U_n\,) \underset{D}{=} \frac{1}{n}\cdot\left\lfloor\frac{n}{p}\right\rfloor \approx \frac{1}{p}$$

(NOTE: $(Y_{n,p})_{p\in\mathbb{P}}$ ARE NOT INDEPENDENT)

TRICK: TRUNCATION. LET US DEFINE

$\boxed{T_n \underset{E}{=} \sum_{\substack{p\in\mathbb{P}\\ p\le \alpha_n}} Y_{n,p}}$ WHERE $\boxed{\alpha_n \underset{F}{:=} n^{1/\ln(\ln(n))}}$

ENOUGH TO SHOW:

$$(\star) \longrightarrow \boxed{\frac{T_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \underset{G}{\Rightarrow} \mathcal{N}(0,1)}$$

BECAUSE: $\ln(\alpha_n) \underset{H}{=} \ln(n)/\ln(\ln(n))$

$$\ln(\ln(\alpha_n)) \underset{I}{=} \ln(\ln(n)) - \ln(\ln(\ln(n)))$$

THUS $T_n \le Z_n$ AND

$$\mathbb{E}(Z_n - T_n) \underset{J}{\le} \sum_{\alpha_n < p \le n} \frac{1}{p} \underset{K}{=} \ln(\ln(\ln(n))) + \mathcal{O}(1)$$

BY CLAIM FROM PAGE 129

THUS $\frac{Z_n - T_n}{\sqrt{\ln(\ln(n))}} \underset{A}{\Longrightarrow} 0$ BY MARKOV'S INEQ.

HENCE IF WE PROVE $(\star)$ THEN ☺ WILL FOLLOW.

LET $(X_p)_{p \in P}$ BE <u>INDEP.</u>, $\boxed{X_p \underset{B}{\sim} \mathrm{BER}\left(\frac{1}{p}\right)}$

LET $\boxed{S_n \underset{C}{:=} \sum_{\substack{p \in P \\ p \leq \alpha_n}} X_p}$

<u>LEMMA</u>: $\boxed{\frac{S_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}} \underset{D}{\Longrightarrow} \mathcal{N}(0,1)}$

<u>PROOF</u>: ENOUGH TO SHOW:

$$\boxed{\frac{S_n - \ln(\ln(\alpha_n))}{\sqrt{\ln(\ln(\alpha_n))}} \underset{E}{\Longrightarrow} \mathcal{N}(0,1)} \leftarrow ☹$$

SINCE $\ln(\ln(\alpha_n)) \underset{F}{=} \ln(\ln(n)) - \ln(\ln(\ln(n)))$

WE WILL USE LINDEBERG TO PROVE ☹.

$$\mathbb{E}(S_n) \underset{G}{=} \sum_{\substack{p \in P \\ p \leq \alpha_n}} \frac{1}{p} \underset{H}{=} \ln(\ln(\alpha_n)) + O(1)$$

$$\mathrm{Var}(S_n) \underset{I}{=} \sum_{\substack{p \in P \\ p \leq \alpha_n}} \left(\frac{1}{p} - \frac{1}{p^2}\right) \underset{J}{=} \ln(\ln(\alpha_n)) + O(1)$$

LINDEBERG'S CONDITION:

LET $\tilde{X}_p := X_p - \mathbb{E}(X_p)$ A THEN $\boxed{|\tilde{X}_p| \leq 1}$ B

GIVEN SOME $\varepsilon > 0$:

$$\lim_{n \to \infty} \frac{1}{\ln(\ln(d_n))} \sum_{\substack{p \in P \\ p \leq d_n}} \mathbb{E}\left(|\tilde{X}_p|^2 \cdot \mathbb{1}\left[|\tilde{X}_p| > \varepsilon \cdot \sqrt{\ln(\ln(d_n))} \right] \right) = 0$$

C

SINCE $\varepsilon \cdot \sqrt{\ln(\ln(d_n))} > 1$ D IF n IS LARGE ENOUGH

✓✓✓

LET $\boxed{S_n^* := \frac{S_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}}}$ E $\boxed{T_n^* = \frac{T_n - \ln(\ln(n))}{\sqrt{\ln(\ln(n))}}}$ F

$\boxed{\underline{\text{KNOW}}: \; S_n^* \Rightarrow \mathcal{N}(0,1)}$ G $\boxed{\underline{\text{WANT}}: \; T_n^* \Rightarrow \mathcal{N}(0,1)}$ H

ACTUALLY WE KNOW MORE (HW 9.1):

$$\forall r \in \mathbb{N}: \lim_{n \to \infty} \mathbb{E}\left((S_n^*)^r\right) = \mathbb{E}(X^r), \quad \underbrace{X \sim \mathcal{N}(0,1)}$$

I

WE WILL NOW SHOW:

$$\forall r \in \mathbb{N}: \lim_{n \to \infty} \mathbb{E}\left((T_n^*)^r\right) = \mathbb{E}(X^r), \quad -\|-$$

J

LEMMA: $$\left|\mathbb{E}(S_m^r) - \mathbb{E}(T_m^r)\right| \underset{A}{\leq} \frac{1}{m}\cdot(\alpha_m)^r$$

PROOF: LET $$\boxed{N \underset{B}{:=} \sum_{\substack{p\in\mathcal{P}\\ p\leq \alpha_m}} 1}$$ $$\boxed{\text{NOTE: } N \underset{C}{\leq} \alpha_m}$$

$$(X_1+\dots+X_N)^r \underset{D}{=}$$

$$\underbrace{\sum_{l=1}^{N} \sum_{\substack{r_1,\dots,r_l\geq 1\\ r_1+\dots+r_l=r}} \sum_{1\leq m_1<m_2<\dots<m_l\leq N} \overbrace{C(l;r_1,\dots,r_l)}^{\geq 0}} \cdot X_{m_1}^{r_1}\cdot\ldots\cdot X_{m_l}^{r_l}$$

THUS: $$\mathbb{E}(S_m^r) \underset{E}{=} -\|- \cdot \mathbb{E}\left(\mathbb{X}_{m_1}^{r_1}\cdot\ldots\cdot\mathbb{X}_{m_l}^{r_l}\right)$$

NOW IF $1 < p_1 < p_2 < \dots < p_l \leq \alpha_m$ THEN

$$\mathbb{E}\left(\mathbb{X}_{p_1}^{r_1}\cdot\ldots\cdot\mathbb{X}_{p_l}^{r_l}\right) \underset{F}{=} \mathbb{E}\left(\mathbb{X}_{p_1}\cdot\ldots\cdot\mathbb{X}_{p_l}\right) \underset{G}{=} \frac{1}{p_1\cdot p_2\cdot\ldots\cdot p_l}$$

$$\mathbb{E}\left(\mathbb{Y}_{m,p_1}^{r_1}\cdot\ldots\cdot\mathbb{Y}_{m,p_l}^{r_l}\right) \underset{H}{=} \mathbb{E}\left(\mathbb{Y}_{m,p_1}\cdot\ldots\cdot\mathbb{Y}_{m,p_l}\right) \underset{I}{=} \frac{1}{m}\cdot\left\lfloor\frac{m}{p_1\cdot p_2\cdot\ldots\cdot p_l}\right\rfloor$$

THUS:

$$0 \underset{J}{\leq} \mathbb{E}(S_m^r) - \mathbb{E}(T_m^r) \underset{K}{\leq} \sum_{l=1}^{N} \sum_{r_1,\dots,r_l} \sum_{\dots} C(l;r_1,\dots,r_l)\cdot\frac{1}{m} \underset{L}{=}$$

$$= \frac{1}{m}\cdot(1+1+\dots+1)^r = \frac{1}{m}\cdot N^r \underset{M}{\leq} \frac{1}{m}\cdot(\alpha_m)^r$$

COROLLARY:

$$\forall r \in \mathbb{N}: \quad \left| \mathbb{E}\left[(S_n^*)^r \right] - \mathbb{E}\left[(T_n^*)^r \right] \right| \underset{A}{\xrightarrow[n \to \infty]{}} 0$$

PROOF:

$$\mathbb{E}\left[(S_n^*)^r \right] \underset{B}{=} \frac{1}{\ln(\ln(n))^{r/2}} \mathbb{E}\left[\left(S_n - \ln(\ln(n)) \right)^r \right]$$

$$\underset{C}{=} \frac{1}{\ln(\ln(n))^{r/2}} \sum_{l=0}^{r} \binom{r}{l} \cdot \mathbb{E}(S_n^l) \cdot \left(-\ln(\ln(n)) \right)^{r-l}$$

THUS $\left| \mathbb{E}((S_n^*)^r) - \mathbb{E}((T_n^*)^r) \right| \underset{D}{=}$

$$\left| \frac{1}{\ln(\ln(n))^{r/2}} \sum_{l=0}^{r} \binom{r}{l} \cdot \left(\mathbb{E}(S_n^l) - \mathbb{E}(T_n^l) \right) \cdot \left(-\ln(\ln(n)) \right)^{r-l} \right|$$

$$\underset{E}{\leq} \sum_{l=0}^{r} \binom{r}{l} \cdot \frac{1}{n} \cdot d_n^l \cdot \ln(\ln(n))^{r-l} \underset{F}{=}$$

$$= \frac{1}{n} \cdot \left(d_n \cdot \ln(\ln(n)) \right)^r \underset{G}{\xrightarrow[n \to \infty]{}} 0$$

BECAUSE $d_n = n^{1/\ln(\ln(n))}$, THUS $\forall \varepsilon > 0$

$d_n \underset{H}{\leq} n^{\varepsilon}$ IF n IS LARGE ENOUGH.

THUS

$$\forall r \in \mathbb{N}: \boxed{\lim_{n\to\infty} \mathbb{E}\left(\left(T_n^*\right)^r\right) = \mathbb{E}\left(X^r\right)}, \quad X \sim \mathcal{N}(0,1)$$

IT REMAINS TO SHOW THAT THIS IMPLIES $\boxed{T_n^* \Longrightarrow \mathcal{N}(0,1)}$

THE MOMENT PROBLEM:

IF $\forall r \in \mathbb{N}: \mathbb{E}(V^r) = \mathbb{E}(W^r) < +\infty$

DOES THIS IMPLY THAT $\boxed{V \sim W}$?

UNFORTUNATELY NOT, E.G. THE LOG-NORMAL DISTRIBUTION IS NOT DETERMINED BY ITS MOMENTS (HW 9.2). HOWEVER:

LEMMA: IF $M_r = \mathbb{E}(V^r) = \mathbb{E}(W^r)$ AND

$$\limsup_{n\to\infty} \left(\frac{|M_r|}{r!}\right)^{1/r} =: R^{-1} < +\infty \quad \text{THEN } \boxed{V \sim W}.$$

PROOF: SEE NEXT PAGE

$\underline{\text{PROOF}}$: $\varphi(t) \underset{A}{=} \mathbb{E}\left(e^{it \cdot V}\right) \underset{B}{=} \sum_{r=0}^{\infty} \frac{M_r}{r!} i^r \cdot t^r$

THIS POWER SERIES WILL HAVE A $\underline{\text{RADIUS}}$ $\underline{\text{OF}}$ $\underline{\text{CONVERGENCE}}$ $R > 0$, THUS $\varphi(t)$ UNIQUELY EXTENDS AS AN ANALYTIC FUNCTION TO THE STRIP $\{t \in \mathbb{C} : \Im m(t) \in (-R, R)\}$, SEE PAGE 85.

THUS $\boxed{\varphi_V(t) \underset{C}{\equiv} \varphi_W(t)}$, THUS $\boxed{V \underset{D}{\sim} W}$

$\underline{\text{EX}}$: $\mathcal{N}(0,1)$: $M(\lambda) \underset{E}{=} e^{\lambda^2/2} \underset{F}{=} \sum_{\ell=0}^{\infty} \frac{(\lambda^2/2)^\ell}{\ell!}$

THUS $\sum_{r=0}^{\infty} \frac{M_r}{r!} \lambda^r \underset{G}{=} \sum_{\ell=0}^{\infty} \frac{\lambda^{2\ell}}{2^\ell \cdot \ell!}$, THUS

RADIUS OF CONVERGENCE: $R = +\infty$

$M_r = 0$ IF r IS ODD AND

IF $r = 2\ell$ THEN $M_r = M_{2\ell} \underset{H}{=} \frac{(2\ell)!}{2^\ell \cdot \ell!}$

THM: LET $M_{n,r} \underset{A}{:=} \mathbb{E}\left((V_n)^r\right) < +\infty$

IF $\forall r \in \mathbb{N}$: $\boxed{\lim_{n\to\infty} M_{n,r} \underset{B}{=:} M_r}$ AND

THE SEQUENCE OF MOMENTS $(M_r)_{r=0}^{\infty}$

UNIQUELY DETERMINES A DISTRIBUTION

(RANDOM VARIABLE): V, THEN $\boxed{V_n \underset{C}{\Rightarrow} V}$

PROOF: $(V_n)_{n=1}^{\infty}$ IS A TIGHT SEQUENCE:

$$\mathbb{P}(|V_n| \geq K) \underset{D}{=} \mathbb{P}(V_n^2 \geq K^2) \underset{E}{\leq} \frac{M_{n,2}}{K^2}$$

MARKOV

THUS GIVEN SOME $\varepsilon > 0$ WE HAVE

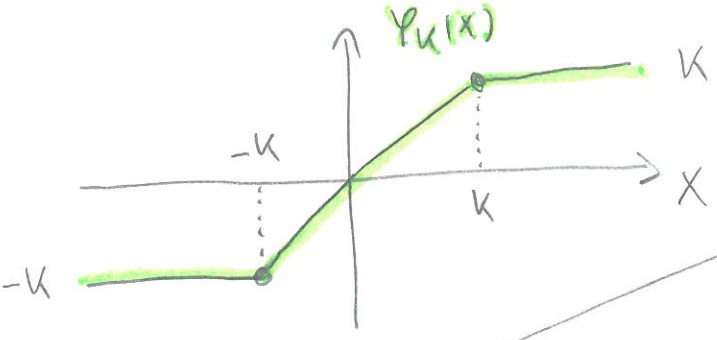
$$\sup_n \mathbb{P}(|V_n| \geq K) \underset{F}{\leq} \varepsilon \quad \text{IF} \quad K \underset{G}{:=} \sqrt{\frac{\sup_n M_{n,2}}{\varepsilon}}$$

NOW WE SHOW THAT IF $V_{n'}$ IS A SUBSEQUENCE WHICH CONVERGES IN DISTRIBUTION $(V_{n'} \underset{H}{\Rightarrow} \tilde{V})$ THEN

$\boxed{\mathbb{E}(\tilde{V}^r) \underset{I}{=} M_r}$, $r \in \mathbb{N}$

INDEED: LET US DEFINE

$$\varphi_K(x) \underset{A}{=} x\cdot \mathbb{1}[|x|\le K] + \mathrm{sign}(x)\cdot K\cdot \mathbb{1}[|x|>K]$$



IF r IS EVEN: BY MONOTONE CONV.

IF r IS ODD: DOMINATED CONV.

$$\mathbb{E}(\tilde V^r) \underset{B}{=} \lim_{K\to\infty} \mathbb{E}\left(\varphi_K(\tilde V)^r\right)$$

SINCE $V_{n'} \Rightarrow \tilde V$, φ_K IS BOUNDED CONTINUOUS

$$\underset{C}{=} \lim_{K\to\infty} \lim_{n'\to\infty} \mathbb{E}\left(\varphi_K(V_{n'})^r\right)$$

$$\underset{D}{=} \lim_{K\to\infty} \lim_{n'\to\infty} \left(\mathbb{E}(V_{n'}^r) - \mathbb{E}\left(V_{n'}^r - \varphi_K(V_{n'})^r\right)\right)$$

$$\underset{E}{=} \underbrace{\lim_{n'\to\infty} M_{n',r}}_{M_r \; F} - \underbrace{\lim_{K\to\infty} \lim_{n'\to\infty} \mathbb{E}\left(V_{n'}^r - \varphi_K(V_{n'})^r\right)}_{G}$$

WE WILL SHOW THAT THIS IS ZERO

$$\left|\mathbb{E}\left(V_n^r - \varphi_K(V_n)^r\right)\right| \underset{H}{\le} \mathbb{E}\left(|V_n|^r \cdot \mathbb{1}[|V_n| \ge K]\right)$$

$$\underset{I}{\le} \sqrt{\mathbb{E}(V_n^{2r})} \cdot \sqrt{\mathbb{E}\left(\mathbb{1}[|V_n|\ge K]^2\right)} \underset{J}{=} \sqrt{M_{n,2r}} \cdot \sqrt{\mathbb{P}(|V_n| \ge K)} \underset{K}{=} ☺$$

CAUCHY-SCHWARZ

$$\text{☺} \underset{A}{\leq} \sqrt{M_{n,2r}} \cdot \sqrt{M_{n,2}/K^2} \underset{B}{\leq} \frac{\sqrt{\sup_n M_{n,2r} \cdot M_{n,2}}}{K}$$

↑ MARKOV INEQ

THUS $\text{☹} \underset{C}{\leq} \lim_{K \to \infty} \frac{\sqrt{\sup_n M_{n,2r} \cdot M_{n,2}}}{K} \underset{D}{=} 0$ ✓

THUS $\mathbb{E}(\tilde{V}^r) \underset{E}{=} M_r, \; \forall r \in \mathbb{N}$, THUS $\boxed{\tilde{V} \underset{F}{\sim} V}$

TO SHOW THAT THE WHOLE SEQUENCE V_n CONVERGES IN DISTRIBUTION TO V, JUST REPEAT THE ARGUMENT FROM PAGE 110–112.