RECALL: CHARACTERISTIC FUNCTION OF $\mathbb{X}$:

$$\Psi_{\mathbb{X}}(t) \underset{A}{=} \mathbb{E}\left(e^{i\cdot t\cdot \mathbb{X}}\right), \quad t \in \mathbb{R}$$

IF $\mathbb{X}$ HAS A P.D.F. f, THEN $\Psi_{\mathbb{X}}(t) \underset{B}{=} \int_{-\infty}^{\infty} e^{i\cdot t\cdot x}\cdot f(x)\,dx$

THM: IF $\boxed{\Psi_{\mathbb{X}}(t) \underset{C}{=} \Psi_{\mathbb{Y}}(t) \quad \forall t \in \mathbb{R}}$ THEN $\boxed{\mathbb{P}(\mathbb{X} \le x) \underset{D}{=} \mathbb{P}(\mathbb{Y} \le x) \quad \forall x \in \mathbb{R}}$

(I.E., CHAR. FN. INDEED CHARACTERIZES THE DISTRIBUTION)

INGREDIENTS OF PROOF:

FACT: (A SPECIAL CASE OF FUBINI'S THM)

$\boxed{g : \mathbb{R}^2 \to \mathbb{C}}$, IF $\mathbb{E}\left(\int_{-\infty}^{\infty} |g(\mathbb{X}, t)|\,dt\right) \underset{E}{<} +\infty$

THEN $\boxed{\mathbb{E}\left(\int_{-\infty}^{\infty} g(\mathbb{X}, t)\,dt\right) \underset{F}{=} \int_{-\infty}^{\infty} \mathbb{E}\left(g(\mathbb{X}, t)\right) dt}$

RECALL: $\varphi_a(x) \underset{G}{=} \frac{1}{a\cdot\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2\cdot a^2}\right)$ ← $\boxed{\text{P.D.F. OF } \mathcal{N}(0, a^2)}$

THEN $\int_{-\infty}^{\infty} e^{i\cdot t\cdot x}\cdot \varphi_a(x)\,dx \underset{H}{=} e^{-\frac{1}{2}a^2\cdot t^2}$

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THUS $\int_{-\infty}^{\infty} e^{i\cdot t\cdot x}\cdot \varphi_a(t)\,dt \underset{A}{=} e^{-\frac{1}{2}\cdot a^2\cdot x^2}$, THUS

$$\boxed{\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\cdot t\cdot(y-x)}\cdot e^{-t^2\cdot\sigma^2/2}\,dt \underset{B}{=}\\ &= \frac{1}{\sigma\cdot\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\cdot t\cdot(y-x)}\cdot \varphi_{1/\sigma}(t)\,dt \underset{C}{=} \frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-y)^2}{2\sigma^2}\right)\end{aligned}}$$

LET $\mathbb{Y}\sim\mathcal{N}(0,1)$, INDEP. OF $\mathbb{X}$

LET $\mathbb{X}_\sigma \underset{D}{:=} \mathbb{X}+\sigma\cdot\mathbb{Y}$, $F_\sigma(x) \underset{E}{:=} \mathbb{P}(\mathbb{X}_\sigma\le x)$

$$F_\sigma(x)\underset{F}{=}\mathbb{P}\left(\mathbb{Y}\le \frac{x-\mathbb{X}}{\sigma}\right) \underset{G}{=} \mathbb{E}\left(\Phi\left(\frac{x-\mathbb{X}}{\sigma}\right)\right)$$

$$f_\sigma(x) := \frac{d}{dx}F_\sigma(x) \underset{H}{=} \mathbb{E}\left(\frac{d}{dx}\Phi\left(\frac{x-\mathbb{X}}{\sigma}\right)\right) \underset{I}{=}$$

$$= \mathbb{E}\left(\frac{1}{\sigma}\varphi\left(\frac{x-\mathbb{X}}{\sigma}\right)\right) \underset{J}{=} \mathbb{E}\left(\varphi_\sigma(x-\mathbb{X})\right) \underset{K}{=} \mathbb{E}\left(\frac{1}{\sigma\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-\mathbb{X})^2}{2\sigma^2}\right)\right)$$

$$\psi_\sigma(t) \underset{L}{=} \mathbb{E}\left(e^{i\cdot t\cdot \mathbb{X}_\sigma}\right) \underset{M}{=} \mathbb{E}\left(e^{it\mathbb{X}}\cdot e^{it\sigma\cdot\mathbb{Y}}\right) \underset{N}{=}$$

$$= \psi(t)\cdot e^{-\frac{1}{2}\sigma^2\cdot t^2}$$

LEMMA (INVERSION FORMULA FOR SMOOTHED DISTRIBUTION)

$$\boxed{f_\sigma(x) \underset{A}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\cdot t\cdot x}\cdot \psi_\sigma(t)\,dt} \qquad x\in\mathbb{R}$$

PROOF: LET US FIX $x\in\mathbb{R}$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot\psi_\sigma(t)\,dt \underset{B}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot e^{-\frac{t^2\cdot\sigma^2}{2}}\cdot \mathbb{E}\left(e^{itX}\right)dt \underset{C}{=}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}\Big(\underbrace{e^{it\cdot(X-x)}\cdot e^{-\frac{t^2\cdot\sigma^2}{2}}}_{g(X,t)}\Big)\,dt \overset{\text{FUBINI}}{\underset{D}{=}} ☺$$

FUBINI IS APPLICABLE:

$$\mathbb{E}\left(\int_{-\infty}^{\infty}|g(X,t)|\,dt\right) \underset{E}{=} \mathbb{E}\left(\int_{-\infty}^{\infty} e^{-\frac{t^2\cdot\sigma^2}{2}}\,dt\right) \overset{F}{<} +\infty$$

$$☺ = \mathbb{E}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\cdot(X-x)}\cdot e^{-t^2\cdot\sigma^2/2}\,dt\right) \overset{\text{☆ FROM PAGE 102}}{\underset{G}{=}}$$

$$\mathbb{E}\left(\frac{1}{\sigma\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-X)^2}{2\sigma^2}\right)\right) \underset{H}{=} f_\sigma(x)$$

✓

FACT: IF X HAS A CONTINUOUS P.D.F. $f(x)$

THEN $f(x) \underset{A}{=} \lim\limits_{\sigma\to 0} f_\sigma(x) \underset{B}{=} \lim\limits_{\sigma\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot e^{-\frac{1}{2}\sigma^2t^2}\cdot \psi(t)\,dt$

PROOF: $\boxed{\text{HOMEWORK } 8.1}$

CLAIM: IF X HAS A CONTINUOUS P.D.F. $f(x)$ AND

IF $\int_{-\infty}^{\infty} |\psi(t)|\,dt \underset{C}{<} +\infty$ THEN $\boxed{f(x) \underset{D}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot\psi(t)\,dt}$

PROOF: $\underset{E}{}\lim\limits_{\sigma\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\cdot e^{-\frac{1}{2}\sigma^2t^2}\cdot\psi(t)\,dt \underset{F}{=}$ —||—

BY DOMINATED CONVERGENCE THM:

$\forall \sigma > 0 : \left| e^{-itx}\cdot e^{-\frac{1}{2}\sigma^2\cdot t^2}\cdot\psi(t)\right| \underset{G}{\leq} |\psi(t)|$

PROOF OF THM STATED ON PAGE 101:

GIVEN $\psi(t), t\in\mathbb{R}$, WE CAN RECOVER

$F_\sigma(x) \underset{H}{=} \int_{-\infty}^{x} f_\sigma(u)\,du$ BY LEMMA FROM PAGE 103,

AND $\boxed{F_\sigma \Rightarrow F}$ AS $\boxed{\sigma\to 0}$ BY SLUTSKY. THUS

WE CAN RECOVER F FROM ψ. ✓

DEF: CAUCHY DISTRIBUTION

$\boxed{X \underset{A}{\sim} CAU(1)}$ IF $\boxed{F(x) \underset{B}{=} \frac{1}{2} + \frac{1}{\pi} \cdot \arctan(x)}$

THUS $\boxed{f(x) \underset{C}{=} F'(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}}$

$\boxed{X \underset{D}{\sim} CAU(a)}$ IF $\boxed{\frac{X}{a} \underset{E}{\sim} CAU(1)}$

NOTE: IF $X \sim CAU(1)$ THEN $\mathbb{E}(|X|) \underset{F}{=} +\infty$

INDEED: $\underset{G}{}\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi} \cdot \frac{1}{1+x^2} \, dx \underset{H}{=} 2 \cdot \int_0^{\infty} x \cdot \frac{1}{\pi} \cdot \frac{1}{1+x^2} \, dx \underset{I}{\geq}$

$$\geq \frac{1}{\pi} \int_1^{\infty} \frac{1}{x} \, dx = +\infty$$

$$Z(\lambda) = \mathbb{E}(e^{\lambda X}) \underset{J}{=} \begin{cases} 1 & \text{IF } \lambda = 0 \\ +\infty & \text{IF } \lambda \in \mathbb{R} \setminus \{0\} \end{cases}$$

CLAIM: IF $X \sim CAU(1)$ THEN $\boxed{\varphi(t) \underset{K}{=} e^{-|t|}}$

PROOF: WE KNOW FROM $\boxed{\text{HW } 6.2(a)}$ THAT

$$\boxed{\int_{-\infty}^{\infty} e^{itx} \cdot \frac{1}{2} e^{-|x|} \, dx \underset{L}{=} \frac{1}{1+t^2}}$$

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NOW $x \mapsto \frac{1}{2} e^{-|x|}$ IS CONTINUOUS AND

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt \underset{A}{<} +\infty$$ THUS BY CLAIM ON PAGE 104

WE HAVE $$\frac{1}{2} e^{-|x|} \underset{B}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \cdot \frac{1}{1+t^2}\, dt$$

THUS $$\int_{-\infty}^{\infty} e^{itx} \cdot \frac{1}{\pi} \cdot \frac{1}{1+x^2}\, dx \underset{C}{=} e^{-|-t|} = e^{-|t|} \quad \checkmark$$

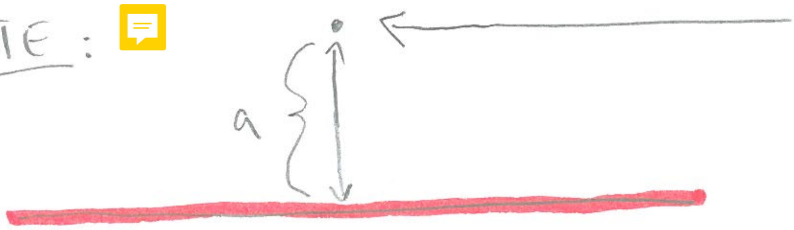
THUS: $$\boxed{X \sim CAU(a)} \underset{D}{\Longleftrightarrow} \boxed{\mathbb{E}\left(e^{itX}\right) = e^{-a\cdot|t|}}$$

CLAIM: IF $X \sim CAU(a)$, $Y \sim CAU(b)$, INDEP.

THEN $$\boxed{(X+Y) \underset{E}{\sim} CAU(a+b)}$$

PROOF: $$\varphi_{X+Y}(t) \underset{F}{=} \varphi_X(t) \cdot \varphi_Y(t) \underset{G}{=} e^{-a\cdot|t|} \cdot e^{-b\cdot|t|} = e^{-(a+b)\cdot|t|}$$

NOTE:

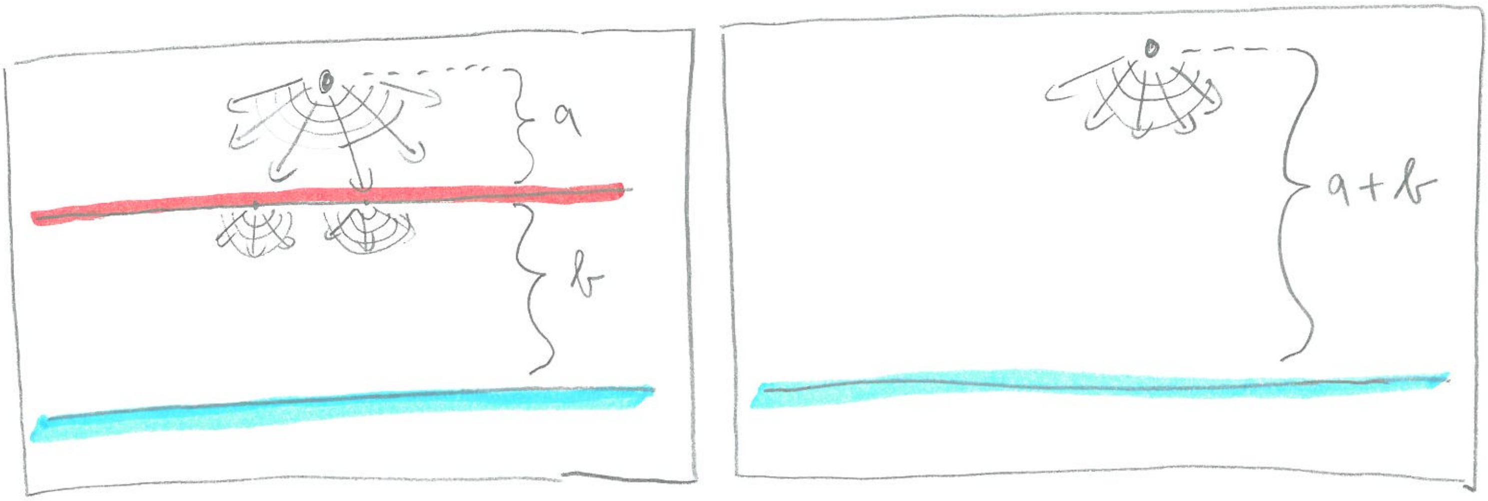


A LIGHTBULB THAT EMITS UNIT AMOUNT OF LIGHT

THEN THE LIGHT INTENSITY DISTRIBUTION ON THE RED LINE IS $CAU(a)$

(EASY TO CHECK USING DEF OF arctan FUNCTION)

THUS: DIFFRACTION (HUYGENS-FRESNEL PRINCIPLE)



ON THE LEFT PICTURE, THE RED LINE IS MADE OF „MILK GLASS", I.E., LIGHT IS „DIFFUSED". ON BOTH PICTURES, THE LIGHT DISTRIBUTION ON BLUE LINE IS CAU(a+b).

ON THE LEFT, WE USED THAT THE CONVOLUTION OF CAU(a) AND CAU(b) IS CAU(a+b).

LAW OF LARGE NUMBERS FAILS FOR CAUCHY DISTRIBUTION:

IF $X_1, X_2, \ldots$ I.I.D. CAU(1), THEN

$S_n = X_1 + \ldots + X_n$, $S_n \sim CAU(n)$, THUS

$$\boxed{\frac{S_n}{n} \sim CAU(1)}$$

THUS $\frac{S_n}{n}$ DOES NOT CONVERGE TO A DETERMINISTIC NUMBER AS $n \to \infty$.