RECALL: (X_n) SIMPLE R.W. ON $\mathbb{Z}$

$$\Pi_n := \#\{ j \in (0,n] : X_{j-1} + X_j > 0 \}$$ A

(TIME SPENT ON POSITIVE HALF-LINE BY TIME n)

THM. (PAUL LÉVY'S ARCSINE THM):

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\Pi_{2n}}{2n} \le x \right) \overset{B}{=} \begin{cases} 0 \text{ IF } x \le 0 \\ \frac{2}{\pi} \cdot \arcsin(\sqrt{x}), \ 0 \le x \le 1 \\ 1 \text{ IF } x \ge 1 \end{cases}$$

THE ONLY THING LEFT TO PROVE:

MAGIC LEMMA: $k = 0, 1, 2, \dots, n$

$$\mathbb{P}(\Pi_{2n} = 2k) \overset{C}{=} u(2k) \cdot u(2 \cdot (n-k)), \text{ WHERE}$$

$$\boxed{u(2k) \underset{D}{=} \mathbb{P}(X_{2k} = 0)}$$

NOTE: SYMMETRY:

$$\mathbb{P}(\Pi_{2n} = 2k) = \mathbb{P}(\Pi_{2n} = 2(n-k))$$

E ✓✓✓

FIRST LET'S PROVE IT WHEN $k=0$ (OR $k=n$):

FACT: $$\mathbb{P}(\Pi_{2n} = 0) \underset{F}{=} u(2n) \quad \Big(= \underbrace{u(2\cdot 0)}_{=1} \cdot u(2\cdot(n-0))\Big)$$ G

PROOF: $$\mathbb{P}(\Pi_{2n} = 0) \underset{H}{=} \mathbb{P}(M_{2n} = 0) \underset{I}{=} \mathbb{P}(T_1 > 2n) \overset{J}{=}$$

$$= \mathbb{P}(X_{2n} = 0) + \mathbb{P}(X_{2n} = 1) \underset{K}{=} u(2n) + 0 = u(2n)$$

SEE THE SOLUTION OF $\boxed{\text{HW 5.1}}$ ✓

WE WILL PROVE MAGIC LEMMA BY INDUCTION ON n.

DEF: A $R_1 = \min\{n \geq 1 : X_n = 0\}$

LET B $\boxed{f(r) := \mathbb{P}(R_1 = r)}$

C (NOTE: $f(1) = f(3) = f(5) = \ldots = 0$)

CLAIM: D $u(2n) = \sum_{r=1}^{n} f(2r) \cdot u(2(n-r))$

PROOF: $\{X_{2n} = 0\} \underset{E}{=} \{X_{2n} = 0\} \cap \{R_1 \leq 2n\} =$

$$\underset{G}{=} \bigcup_{r=1}^{n} \{R_1 = 2r,\ X_{2n} = 0\}$$

$$u(2n) = \mathbb{P}(X_{2n} = 0) \underset{H}{=} \sum_{r=1}^{n} \mathbb{P}(R_1 = 2r,\ X_{2n} = 0) =$$

$$\underset{I}{=} \sum_{r=1}^{n} \underbrace{\mathbb{P}(R_1 = 2r)}_{f(2r)} \cdot \underbrace{\mathbb{P}(X_{2(n-r)} = 0)}_{u(2(n-r))}$$

PROOF OF MAGIC LEMMA: (INDUCTION ON n):

BY FACT (PAGE 73), WE MAY J ASSUME $0 < r < n$,

THUS $R_1 \leq 2n$. THE FIRST EXCURSION OF THE R.W. IS EITHER TO THE POSITIVE SIDE OR TO THE NEGATIVE SIDE (WITH $\frac{1}{2}$-$\frac{1}{2}$ K CHANCE),

SO (SEE NEXT PAGE)

SO:

$$\mathbb{P}(\Pi_{2n} = 2k) =$$ A

B $$\frac{1}{2} \sum_{r=1}^{k} f(2r) \cdot \mathbb{P}(\Pi_{2(n-r)} = 2(k-r)) +$$

C $$\frac{1}{2} \sum_{r=1}^{n-k} f(2r) \cdot \mathbb{P}(\Pi_{2(n-r)} = 2k) =$$ D

INDUCTION HYPOTHESIS

$$= \frac{1}{2} \sum_{r=1}^{k} f(2r) \cdot u(2 \cdot (k-r)) \cdot u(2 \cdot (n-k)) +$$ E

CLAIM FROM PAGE 74

F $$\frac{1}{2} \sum_{r=1}^{n-k} f(2r) \cdot u(2k) \cdot u(2 \cdot (n-k-r)) =$$

G $$= \frac{1}{2} \cdot u(2 \cdot (n-k)) \cdot u(2k) + \frac{1}{2} \cdot u(2k) \cdot u(2 \cdot (n-k))$$

$$= u(2k) \cdot u(2 \cdot (n-k)) \quad \checkmark$$

DEF: H $\lambda_n := \max \{ j \in (0,n] : X_j = 0 \}$

(LAST VISIT TO THE ORIGIN BY TIME n)

I THM: (ANOTHER ARCSINE THM BY PAUL LÉVY):

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\lambda_{2n}}{2n} \le x \right) = \begin{cases} 0 & \text{IF } x \le 0 \\ \frac{2}{\pi} \cdot \arcsin(\sqrt{x}), & \\ 1 & \text{IF } x \ge 1 \end{cases}$$ J K

PROOF: SEE NEXT PAGE

A IT IS ENOUGH TO SHOW THAT λ_{2n} HAS THE SAME DISTRIBUTION AS π_{2n}.

WE WILL SHOW THAT $\star$

$$\boxed{\mathbb{P}(\lambda_{2n}=2r) \overset{B}{=} u(2r)\cdot u(2\cdot(n-r)), \quad r=0,1,\dots,n}$$

LEMMA: $\boxed{\mathbb{P}(X_j \neq 0,\ j=1,2,\dots,2n) \overset{C}{=} u(2n)}$

PROOF: D (($2\cdot \mathbb{P}(X_j < 0,\ j=1,2,\dots,2n) \overset{E}{=}$

$= 2\cdot\frac{1}{2}\cdot \mathbb{P}(X_j \le 0,\ j=1,2,\dots,2n-1)$

$\overset{F}{=} \mathbb{P}(X_j \le 0,\ j=1,2,\dots,2n) \overset{G}{=} \mathbb{P}(M_{2n}=0) \overset{H}{=} u(2n)$

PROOF OF $\star$:

$\mathbb{P}(\lambda_{2n}=2r) \overset{I}{=} \mathbb{P}(X_{2r}=0,\ X_j\neq 0,\ j=2r+1,\dots,2n)$

$\overset{J}{=} \underbrace{\mathbb{P}(X_{2r}=0)}_{u(2r)} \cdot \underbrace{\mathbb{P}(X_j \neq 0,\ j=1,2,\dots,2\cdot(n-r))}_{K\ u(2\cdot(n-r))}$

MORE ON THE THEORY OF WEAK CONV:

A

DEF: (TIGHTNESS OF A FAMILY OF PROBAB. DISTRIBUTIONS)

$X_1, X_2, X_3, \ldots$ IS A TIGHT SEQUENCE IF

$$\boxed{\lim_{K \to \infty} \inf_{n} \mathbb{P}(|X_n| \le K) = 1}$$ B1

(IN WORDS: NO MASS ESCAPES TO ∞ AS $n \to \infty$)

ALTERNATIVE, FANCY DEF: B2

$\forall \varepsilon > 0 \ \exists H \subseteq \mathbb{R}$ WHERE H IS COMPACT

AND $\boxed{\mathbb{P}(X_n \in H) \ge 1 - \varepsilon}$, $n = 1, 2, 3, \ldots$ C

LEMMA: D IF $X_n \Rightarrow X$ THEN $(X_n)_{n=1}^{\infty}$ IS TIGHT.

PROOF: GIVEN $\varepsilon > 0$, CHOOSE $\tilde{K} \in \mathbb{R}_+$ SUCH

THAT E $\mathbb{P}(|X| \le \tilde{K}) \ge 1 - \frac{\varepsilon}{2}$ AND

F $\mathbb{P}(X = -\tilde{K}) = \mathbb{P}(X = \tilde{K}) = 0$

THEN $\lim_{n \to \infty} \mathbb{P}(X_n \le \tilde{K}) \overset{G}{=} \mathbb{P}(X \le \tilde{K})$

$\lim_{n \to \infty} \mathbb{P}(X_n < -\tilde{K}) \overset{H}{=} \mathbb{P}(X < -\tilde{K})$

THUS $\exists n_0 : \forall n \geq n_0 :$

$$\left| \mathbb{P}(-\tilde{K} \leq X_n \leq \tilde{K}) - \mathbb{P}(-\tilde{K} \leq X \leq \tilde{K}) \right| \leq \frac{\varepsilon}{2}$$

THUS $\forall n \geq n_0 : \mathbb{P}(|X_n| \leq \tilde{K}) \geq 1 - \varepsilon$

NOW CHOOSE $K \geq \tilde{K}$ SUCH THAT FOR ALL $1 \leq n \leq n_0$ WE HAVE $\mathbb{P}(|X_n| \leq K) \geq 1 - \varepsilon$

<u>THM</u> (HELLY): IF $(X_n)_{n=1}^{\infty}$ IS A TIGHT SEQUENCE THEN THERE EXISTS A SUBSEQUENCE $(n_k)_{k=1}^{\infty}$ SUCH THAT X_{n_k} CONVERGES WEAKLY AS $k \to \infty$.

<u>PROOF</u>: LET $(q_\ell)_{\ell=1}^{\infty}$ BE A DENUMERATION OF $\mathbb{Q}$. DENOTE BY $F_n(x) := \mathbb{P}(X_n \leq x)$.

WE WILL CONSTRUCT A SEQUENCE $(n_k)_{k=1}^{\infty}$ SUCH THAT $\lim_{k \to \infty} n_k = +\infty$ AND

$$\forall q \in \mathbb{Q} : F_{n_k}(q) \text{ CONVERGES AS } k \to \infty.$$

WE WILL USE <u>CANTOR'S</u> <u>DIAGONAL</u> <u>ARGUMENT</u>.

① CHOOSE A SUBSEQUENCE $(n_r^{(1)})$ OF $\mathbb{N}$ SUCH THAT

$F_{n_r^{(1)}}(q_1)$ CONVERGES. LET $n_1 := n_1^{(1)}$.

② CHOOSE A SUBSEQUENCE $(n_r^{(2)})$ OF $(n_r^{(1)})$ S.T.

$F_{n_r^{(2)}}(q_2)$ CONVERGES. LET $n_2 := n_2^{(2)}$

③ CHOOSE A SUBSEQUENCE $(n_r^{(3)})$ OF $(n_r^{(2)})$ S.T.

$F_{n_r^{(3)}}(q_3)$ CONVERGES. LET $n_3 := n_3^{(3)}$

④ ETC... $\boxed{n_r := n_r^{(r)}}$, $r \in \mathbb{N}$

THIS WAY:

A $n_1, n_2, n_3, \ldots$ IS A SUBSEQ. OF $(n_r^{(1)})$

B $n_2, n_3, \ldots$ IS A —"— OF $(n_r^{(2)})$

C $n_3, n_4, \ldots$ IS A —"— OF $(n_r^{(3)})$, ETC.

THUS $F_{n_r}(q)$ CONVERGES FOR ALL $q \in \mathbb{Q}$.

D LET $F(q) := \lim_{r \to \infty} F_{n_r}(q)$, $q \in \mathbb{Q}$

THEN F IS NON-DECREASING AND

$\lim_{q \to +\infty} F(q) = 1$, $\lim_{q \to -\infty} F(q) = 0$ BY TIGHTNESS.

NOW FOR ANY $x \in \mathbb{R}$, LET

$$\boxed{\tilde{F}(x) := \inf_{q > x} F(q)}$$

A

THEN:

B — EXTENDED $\tilde{F}$ IS NON-DECREASING ✓

C — $\boxed{\lim_{x \to \infty} \tilde{F}(x) = 1}$, $\boxed{\lim_{x \to -\infty} \tilde{F}(x) = 0}$ ✓

D — $\tilde{F}$ IS RIGHT-CONTINUOUS:

E $$\tilde{F}(x_+) = \inf_{y > x} \tilde{F}(y) = \inf_{y > x} \inf_{q > y} F(q) = \inf_{q > x} F(q) = \tilde{F}(x)$$

F THUS $\tilde{F}$ IS THE C.D.F. OF A R.V. X.

IT REMAINS TO PROVE THAT $\boxed{F_{n_k} \Rightarrow \tilde{F}}$: G

ENOUGH TO SHOW THAT $\forall \varepsilon > 0$: H

$$\tilde{F}(x - \varepsilon) \leq \liminf_{k \to \infty} F_{n_k}(x) \leq \limsup_{k \to \infty} F_{n_k}(x) \leq \tilde{F}(x + \varepsilon)$$

I LET $x - \varepsilon < \tilde{q} < x$, THEN

$$\tilde{F}(x - \varepsilon) \leq F(\tilde{q}) = \lim_{k \to \infty} F_{n_k}(\tilde{q}) \leq \liminf_{k \to \infty} F_{n_k}(x)$$

SIMILARLY: IF $x < \hat{q} < x + \varepsilon$, THEN

$$\tilde{F}(x + \varepsilon) \geq F(\hat{q}) = \lim_{k \to \infty} F_{n_k}(\hat{q}) \geq \limsup_{k \to \infty} F_{n_k}(x)$$

✓

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