RECALL: SIMPLE R.W. ON $\mathbb{Z}$:

A $\boxed{X_n = Y_1 + \ldots + Y_n}$ $Y_1, Y_2, \ldots$ I.I.D.

B $\mathbb{P}(Y_k = +1) = \mathbb{P}(Y_k = -1) = \frac{1}{2}$

C FACT: (C.L.T.): $n^{-1/2} X_n \Rightarrow X \sim \mathcal{N}(0,1)$

D DEF: $M_n = \max\{X_0, X_1, \ldots, X_n\}$

E FACT: $n^{-1/2} M_n \Rightarrow |X|$, $X \sim \mathcal{N}(0,1)$

F DEF: $T_k := \inf\{n \geq 0 : X_n = k\}$

G FACT: $\frac{T_k}{k^2} \Rightarrow \frac{1}{|X|^2}$, $X \sim \mathcal{N}(0,1)$

H NOTE: THE DISTRIBUTION OF $\frac{1}{|X|^2}$ IS CALLED LÉVY DISTRIBUTION.

I C.D.F. OF LÉVY DISTR: $t \geq 0$

$$F(t) = \mathbb{P}\left(\frac{1}{|X|^2} \leq t\right) = \mathbb{P}\left(\frac{1}{\sqrt{t}} \leq |X|\right) = 2\cdot\left(1 - \Phi\left(\frac{1}{\sqrt{t}}\right)\right)$$

J P.D.F. OF LÉVY: $t \geq 0$

$$f(t) = F'(t) = -2\cdot\varphi\left(\frac{1}{\sqrt{t}}\right)\cdot\left(-\frac{1}{2}\right)\cdot t^{-3/2} = \varphi\left(\frac{1}{\sqrt{t}}\right)\cdot t^{-3/2} =$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{1}{2t}\right)\cdot t^{-3/2}$$

A THUS $f(t) \approx \frac{1}{\sqrt{2\pi}} \cdot t^{-3/2}$ AS $t \to \infty$

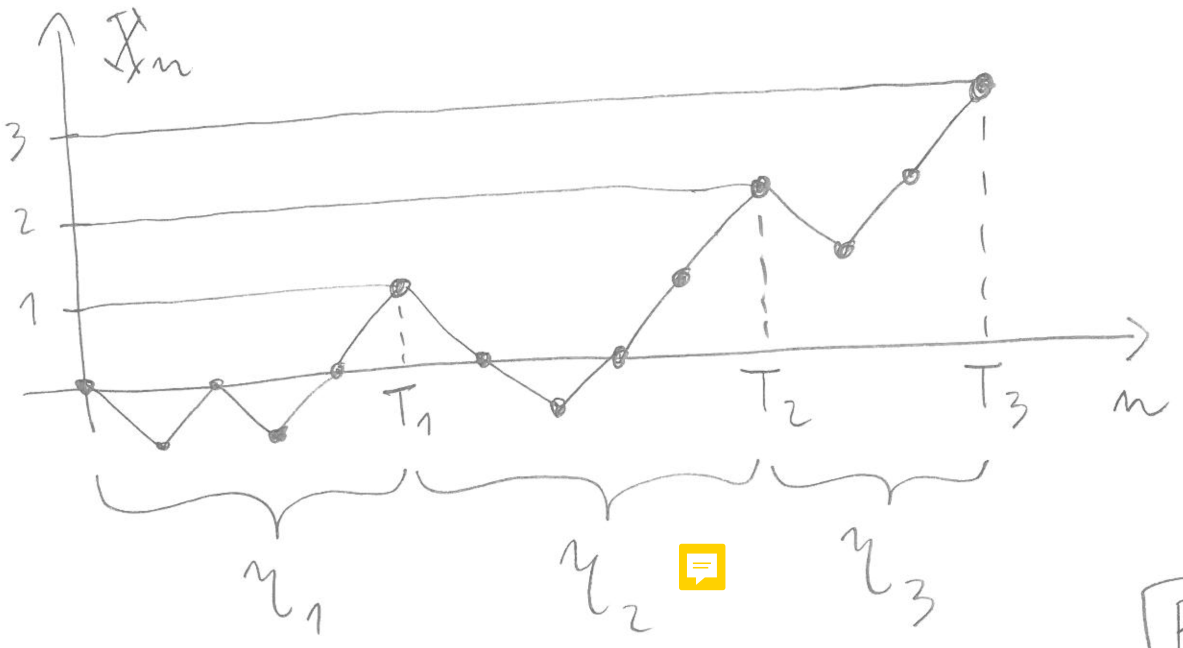
B EXPECTED VALUE OF LÉVY DISTR:

$$\int_0^\infty t \cdot f(t)\, dt \succsim \int_1^\infty t \cdot t^{-3/2}\, dt = +\infty$$

C <u>CLAIM</u>: IF $\eta_1, \eta_2, \dots \eta_k$ ARE I.I.D. WITH THE SAME DISTRIBUTION AS T_1, THEN

D $$\boxed{T_k \sim \eta_1 + \eta_2 + \dots + \eta_k}$$

<u>PROOF</u>: IN ORDER TO HIT LEVEL k, FIRST YOU NEED TO HIT LEVEL 1, THEN YOU RESTART YOUR CLOCK (BY STRONG MARKOV PROP.) AND THEN WAIT UNTIL YOU HIT LEVEL 2, ETC., ... HIT LEVEL k:



THUS IF $\eta_1, \eta_2, \ldots$ I.I.D., $\eta_k \sim T_1$

A $S_n = \eta_1 + \ldots + \eta_n$, THEN $\boxed{S_n \sim T_n}$,

B THUS $\frac{S_n}{n^2} \Rightarrow$ $\boxed{\text{LÉVY DISTRIBUTION}}$

THIS IS SURPRISING! ONE WOULD NAIVELY EXPECT WEAK LAW OF LARGE NUMBERS:

C $\boxed{\frac{S_n}{n} \Rightarrow \mathbb{E}(T_1)}$, AND THIS INDEED HOLDS, BUT $\boxed{\mathbb{E}(T_1) = +\infty}$

D YOU WILL SHOW IN $\boxed{\text{HW 5.1}}$ THAT

IN FACT, YOU WILL SHOW THAT

E $\boxed{\mathbb{P}(T_1 > n) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}}$ AS $n \to \infty$,

F SO IN FACT, THE LARGEST TERM IN THE SUM S_n IS ALREADY OF ORDER n^2:

G $\tilde{M}_n = \max\{\eta_1, \ldots, \eta_n\}$, THEN $\tilde{M}_n \le S_n$

H $\lim_{n\to\infty} \mathbb{P}\left(\frac{\tilde{M}_n}{n^2} \le x\right) \overset{I}{=} \lim_{n\to\infty} \mathbb{P}\left(T_1 \le \lfloor x \cdot n^2 \rfloor\right)^n \overset{J}{=}$

$= \lim_{n\to\infty} \left(1 - \sqrt{\frac{2}{\pi}} \frac{1}{n \cdot \sqrt{x}}\right)^n \overset{K}{=} \exp\left(-\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{x}}\right)$, THUS

A THUS $\frac{\tilde{M}_n}{n^2} \Rightarrow \boxed{\text{FRÉCHET DISTRIBUTION}}$

(SEE SOLUTION OF $\boxed{\text{HW 4.1}}$) THUS $\tilde{M}_n \approx n^2$

B RECALL: IF X_1, X_2 I.I.D. $N(0,1)$, THEN

C $$\boxed{\frac{X_1 + X_2}{\sqrt{2}} \sim X_3 \sim N(0,1)}$$

D WE WILL NOW SHOW THAT IF X_1, X_2 ARE I.I.D. WITH LÉVY DISTR., THEN

E $$\boxed{\frac{X_1 + X_2}{4} \sim X_3 \sim \boxed{\text{LÉVY DISTR.}}}$$, INDEED:

F $$\underbrace{\frac{\eta_1 + \ldots + \eta_{2n}}{(2n)^2}}_{\substack{G \\ \Rightarrow X_3}} = \frac{1}{4} \cdot \left(\underbrace{\frac{\eta_1 + \ldots + \eta_n}{n^2}}_{\Rightarrow X_1} + \underbrace{\frac{\eta_{n+1} + \ldots + \eta_{2n}}{n^2}}_{\Rightarrow X_2} \right)$$

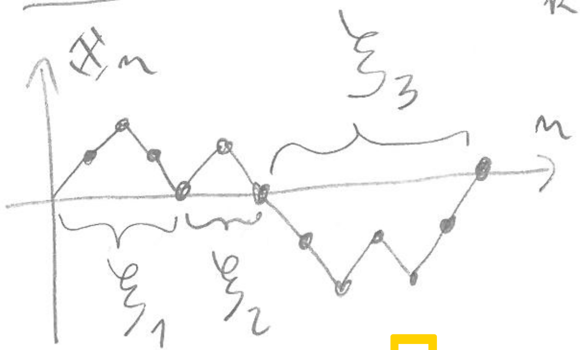
H X_1, X_2 $\boxed{\text{I.I.D.}}$

A **DEF:** R_k IS THE k'TH RETURN TIME OF THE R.W. (X_n) TO THE ORIGIN: $k \in \mathbb{N}$,

B $\boxed{R_0 = 0}$ $\boxed{R_{k+1} := \inf\{n > R_k : X_n = 0\}}$

C **THM:** $\boxed{\frac{R_k}{k^2} \Rightarrow \frac{1}{|X|^2}}$ $X \sim \mathcal{N}(0,1)$ $\leftarrow (\star)$

D **PROOF:** LET $\xi_k := R_k - R_{k-1}$, $k = 1, 2, \dots$



THEN $\xi_1, \xi_2, \dots$ I.I.D.

$\boxed{\xi_k \sim R_1}$

E **NOTE:** $\boxed{R_1 \sim T_1 + 1}$ WHY? YOU MAKE THE FIRST STEP AWAY FROM 0, NOW YOU'RE AT DISTANCE 1. THE TIME IT TAKES TO REACH A TARGET AT DISTANCE 1 HAS THE SAME DISTRIBUTION AS T_1. THUS

F $R_k \sim (\eta_1 + 1) + \dots + (\eta_k + 1) \sim T_k + k$, BUT

G $\boxed{\frac{T_k}{k^2} \Rightarrow \frac{1}{|X|^2}}$ $\boxed{\frac{k}{k^2} \Rightarrow 0}$ SO $(\star)$ FOLLOWS BY SLUTSKY.

$\boxed{\text{PAGE 65}}$

A **DEF: LOCAL TIME OF R.W. AT 0:**

$$L_n := \# \{ j \in (0,n] : X_j = 0 \}$$

$$= \sum_{j=1}^{n} \mathbb{1}[X_j = 0]$$

B $$= \max \{ k : R_k \le n \}$$

L_n IS THE NUMBER OF RETURNS TO THE ORIGIN BY TIME n

C **THM:** $\boxed{n^{-1/2} L_n \Rightarrow |X|}$, $X \sim N(0,1)$

PROOF: NOTE: $\{L_n < k\} \overset{D}{=} \{R_k > n\} =$

$=$ k'TH RETURN TO 0 OCCURED AFTER n

THUS: $\mathbb{P}(n^{-1/2} L_n < x) = \mathbb{P}(L_n < \lfloor \sqrt{n} \cdot x \rfloor) \overset{E}{=}$

$\mathbb{P}(R_{\lfloor \sqrt{n} \cdot x \rfloor} > n) =$

$$\mathbb{P}\left(\frac{R_{\lfloor \sqrt{n} \cdot x \rfloor}}{(\sqrt{n} \cdot x)^2} > \frac{1}{x^2} \right) \xrightarrow[n \to \infty]{F} \mathbb{P}\left(\frac{1}{|X|^2} > \frac{1}{x^2} \right) \overset{G}{=}$$

$= \mathbb{P}(|X| < x)$ ✓

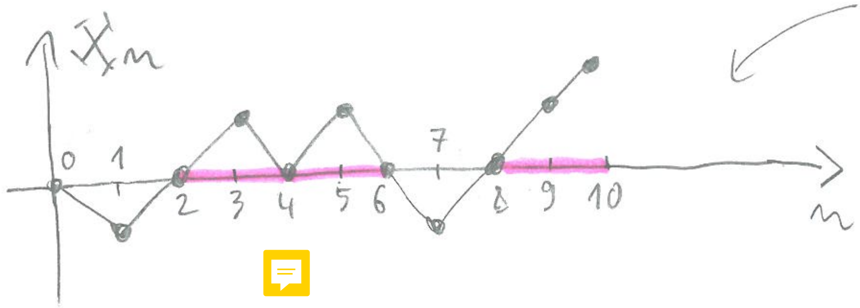
A **CONCLUSION:** $x \geq 0$: $\lim\limits_{n \to \infty} \mathbb{P}\left(n^{-1/2} |X_n| < x \right) =$

B $= \lim\limits_{n \to \infty} \mathbb{P}\left(n^{-1/2} M_n < x \right) = \lim\limits_{n \to \infty} \mathbb{P}\left(n^{-1/2} L_n < x \right) =$

C $= \mathbb{P}\left(|X| \leq x \right) \overset{D}{=} 2 \cdot \Phi(x) - 1$ ✓

E **DEF:** TIME SPENT ON $\mathbb{R}_+$:

$$\Pi_n := \# \{ j \in (0, n] : X_{j-1} + X_j > 0 \}$$



HERE:

$\Pi_2 = 0$, $\Pi_3 = 1$

$\Pi_4 = 2$, $\Pi_9 = 5$

F **QUESTION:** IF $n \gg 1$, DOES THE WALKER SPEND ROUGHLY 50% OF ITS TIME ON $\mathbb{R}_+$?

OR DOES IT SPEND MOST OF ITS TIME ON EITHER THE POSITIVE OR THE NEG. SIDE?

ANSWER: THE TRUTH LIES SOMEWHERE IN BETWEEN:

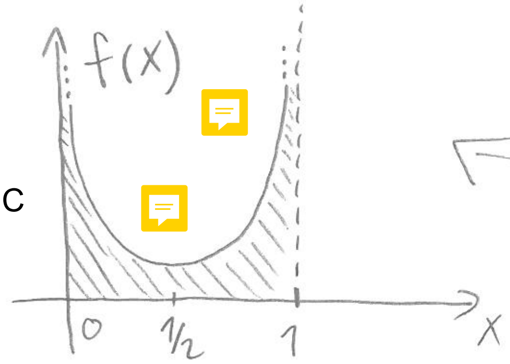
G **THM:** (PAUL LÉVY'S ARCSINE THM):

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\Pi_{2n}}{2n} \leq x \right) = \begin{cases} 0 & \text{IF } x \leq 0 \\ \frac{2}{\pi} \cdot \arcsin(\sqrt{x}), & 0 \leq x \leq 1 \\ 1 & \text{IF } x \geq 1 \end{cases}$$

A **NOTE**: $\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, THUS THE P.D.F. OF LIMITING DISTRIBUTION IS:

B $$\frac{d}{dx}\left(\frac{2}{\pi}\cdot\arcsin(\sqrt{x})\right) = \ldots = \frac{1}{\pi}\cdot\frac{1}{\sqrt{(1-x)\cdot x}} =: f(x)$$

FOR $0<x<1$

C 

AREA UNDER CURVE IS 1.

D **PROOF**:

FIRST NOTE THAT THE POSSIBLE VALUES OF Π_{2n} ARE $0, 2, 4, \ldots, 2n$

E **LEMMA**: (LOCAL LIMIT THM)

$$\lim_{n\to\infty} n\cdot\mathbb{P}\left(\Pi_{2n} = 2\cdot\lfloor n\cdot x\rfloor\right) = \frac{1}{\pi}\cdot\frac{1}{\sqrt{(1-x)\cdot x}}$$

F **PROOF** OF THM FROM PAGE 67 USING LEMMA:

(I.E., PROOF OF GLOBAL USING LOCAL)

LET $Z_n :=$ G $\Pi_{2n} + Y$, WHERE $Y \sim \mathrm{UNI}[0,2]$

AND Y IS INDEP. OF Z_n

A P.D.F. OF Z_{1m} : $f_m(y)$, LET'S FIND IT!

B IF $y \in \mathbb{R}$, THEN $2 \cdot \lfloor y/2 \rfloor$ IS THE LARGEST EVEN NUMBER THAT IS SMALLER THAN OR EQUAL TO y. LET C $\boxed{\tilde{y} := y - 2 \cdot \lfloor y/2 \rfloor}$

$$
\begin{aligned}
f_m(y)\,dy &\overset{D}{=} \mathbb{P}\left(Z_{1m} \in [y, y+dy]\right) \overset{E}{=} \\
&= \mathbb{P}\left(\Pi_{2m} = 2 \cdot \lfloor y/2 \rfloor,\ Y \in [\tilde{y}, \tilde{y}+dy]\right) \overset{\text{INDEP}}{\underset{F}{=}} \\
&= \mathbb{P}\left(\Pi_{2m} = 2 \cdot \lfloor y/2 \rfloor\right) \cdot \underbrace{\mathbb{P}\left(Y \in [\tilde{y}, \tilde{y}+dy]\right)}_{G\quad 1/2\, dy}
\end{aligned}
$$

THUS H $\boxed{f_m(y) = \mathbb{P}\left(\Pi_{2m} = 2 \cdot \lfloor y/2 \rfloor\right) \cdot \frac{1}{2}}$ $\leftarrow$ DENSITY OF Z_{1m}

I <u>SIDE REMARK:</u> IF $F(x) = \mathbb{P}(X \le x)$ AND $Y := \frac{X - a}{b}$ THEN $G(x) \overset{J}{=} \mathbb{P}(Y \le x) = \mathbb{P}(X \le a + bx) = F(a + bx)$. THUS IF P.D.F. OF X IS $f(x) = F'(x)$, THEN P.D.F. OF Y IS $G'(x) = f(a + bx) \cdot b$ H

THUS THE P.D.F. OF $\frac{{}^{+}Z_{1m}}{2m}$ IS $g_m(x)$,

WHERE $g_m(x) = 2m \cdot f_m(2mx) =$

$= 2m \cdot \mathbb{P}\left(\Pi_{2m} = 2 \cdot \lfloor 2mx/2 \rfloor\right) \cdot \frac{1}{2} =$

$= m \cdot \mathbb{P}\left(\Pi_{2m} = 2 \cdot \lfloor mx \rfloor\right) = g_m(x)$

THUS BY THE LOCAL LIMIT THM:

$$\boxed{\lim_{m \to \infty} g_m(x) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{(1-x) \cdot x}}}$$

, THUS BY SCHEFFÉ + SLUTSKY, WE OBTAIN

THAT $\lim_{m \to \infty} \mathbb{P}\left(\frac{\Pi_{2m}}{2m} \leq x\right) = \int_0^x \frac{1}{\pi} \cdot \frac{1}{\sqrt{(1-z) \cdot z}} \, dz$

(I.E., THE GLOBAL LIMIT THM OF PAGE 67)

FOR FURTHER DETAILS, SEE PAGE 55–56.

IT REMAINS TO SHOW THE LOCAL LIMIT THM STATED ON PAGE 68.

A MAGIC LEMMA: $k = 0, 1, 2, \dots, n$

B $\boxed{\mathbb{P}(\pi_{2n} = 2k) = u(2k)\cdot u(2\cdot(n-k))}$, WHERE

C $\boxed{u(k) := \mathbb{P}(X_k = 0)}$ NOTE: IF k IS AN ODD NUMBER, THEN $u(k) = 0$, SINCE THE WALKER CAN ONLY RETURN TO ZERO AFTER MAKING AN EVEN NUMBER OF JUMPS.

D PROOF OF LOCAL LIMIT THM USING MAGIC L:

FIRST NOTE: $\boxed{u(2k) \approx \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{k}}}$ E INDEED:

$$u(2k) = \mathbb{P}(X_{2k} = 0) = \binom{2k}{k}\cdot 2^{-2k} \overset{F}{=} \mathbb{P}(S_{2k} = k),$$

WHERE $S_{2k} \sim BIN(2k, 1/2)$, THUS

BY $\boxed{HW. 4.3}$:

$$\lim_{k\to\infty} \frac{\sqrt{2k}}{2}\,\mathbb{P}\left(S_{2k} = \left\lfloor \frac{2k}{2} + \frac{\sqrt{2k}}{2}\cdot 0 \right\rfloor\right) \overset{G}{=} \frac{1}{\sqrt{2\pi}} e^{-0^2/2}$$

$$\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{2}}\cdot u(2k) \overset{H}{=} \frac{1}{\sqrt{2\pi}} \Rightarrow \lim_{k\to\infty} \sqrt{k\cdot\pi}\cdot u(2k) \overset{I}{=} 1$$

A THUS

$$\lim_{n\to\infty} n \cdot \mathbb{P}\left(\Pi_{2n} = 2\cdot\lfloor n\cdot x\rfloor\right) \stackrel{\text{MAGIC LEMMA}}{=}_{B}$$

$$\lim_{n\to\infty} n \cdot \underbrace{u\left(2\cdot\lfloor n\cdot x\rfloor\right)}_{\approx_{C}\ \boxed{\frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n\cdot x}}}} \cdot \underbrace{u\left(2\cdot\left(n-\lfloor n\cdot x\rfloor\right)\right)}_{\approx_{D}\ \boxed{\frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n\cdot(1-x)}}}} =$$

$$\stackrel{E}{=} \lim_{n\to\infty} n\cdot\frac{1}{\pi}\cdot\frac{1}{n}\cdot\frac{1}{\sqrt{x\cdot(1-x)}} = \frac{1}{\pi}\cdot\frac{1}{\sqrt{x\cdot(1-x)}}$$

WE OBTAINED THE LOCAL LIMIT THM FROM PAGE 68.

F IT REMAINS TO SHOW THE MAGIC LEMMA FROM PAGE 71.