Limit / large dev. thms. exercises before second midterm

1. Prove that the uniform distribution UNI$[-1,1]$ cannot be expressed as the difference of two i.i.d. random variables. *Hint:* Use the method of characteristic functions!

   Solution:(by Joco, approved by Balázs) Let ξ be a RV with distribution $UNI[-1,1]$, then the CF of ξ is:
$$\mathbb{E}(e^{it\xi}) = \int_{-1}^{1} \frac{1}{2} e^{itx} dx = \frac{1}{2it}(e^{it} - e^{-it}) = \frac{sin(t)}{t}.$$
   Suppose that ξ can be expressed as the the difference of two IID random variables: $\xi = X - Y$, where X and Y has CF ψ. Then:
$$\mathbb{E}(e^{it\xi}) = \mathbb{E}(e^{it(X-Y)}) = \mathbb{E}(e^{itX})\mathbb{E}(e^{-itY}) = \psi(t)\overline{\psi(t)} = |\psi(t)|^2$$
   but this can not be the case because $\mathbb{E}(e^{it\xi})$ is not non-negative, but $|\psi(t)|^2$ is. Thus ξ can't be expressed this way.

2. Let X_n be uniformly distributed on the set $\{1, 2, \ldots, n\}$. Use the method of characteristic functions to show that $X_n/n \Rightarrow$ UNI$[0,1]$.

   Solution:(by Joco, approved by Balázs) We will show that the CF of X_n/n converges pointwise to the CF of $X \sim UNI[0,1]$.
$$\mathbb{E}\left(e^{it\frac{X_n}{n}}\right) = \sum_{k=1}^{n} e^{it\frac{k}{n}} \frac{1}{n} = \underbrace{\tfrac{1}{n}\sum_{k=1}^{n} e^{it\frac{k}{n}}}_{B} = \underbrace{\tfrac{1}{n} e^{i\frac{t}{n}} \frac{1-e^{it}}{1-e^{i\frac{t}{n}}}}_{A}$$
   Now there is two way to finish this. The first is that we can say B is a Riemann-sum, thus
$$B \to \int_0^1 e^{itx} dx \quad \text{as } n \to \infty. \tag{1}$$
   where the RHS of (1) is the CF of X. The other way is to calculate $\lim_{n\to\infty} A$:
$$\lim_{n\to\infty} e^{i\frac{t}{n}}(1-e^{it}) \frac{1/n}{1-e^{i\frac{t}{n}}} \overset{L'H}{=} (1-e^{it}) \lim_{n\to\infty} \frac{-1/n^2}{\frac{it}{n^2} e^{it/n}} = \frac{e^{it}-1}{it} \lim_{n\to\infty} \frac{1}{e^{it/n}} = \frac{e^{it}-1}{it}. \tag{2}$$
   where the RHS of (2) is the CF of X.

3. Use the method of characteristic functions to show that the difference of two independent EXP(1) random variables has the same distribution as XY, where $\mathbb{P}(X=1) = \mathbb{P}(X=-1) = \frac{1}{2}$ and $Y \sim$ EXP(1) and X and Y are independent.

   Solution:(by Joco, approved by Balázs) Let Z_1, $Z_2 \sim EXP(1)$ be IID. We will show that XY has the same CF as $Z_1 - Z_2$:
$$\mathbb{E}(e^{itXY}) \underset{\text{tot. prob.}}{\overset{\text{Law of}}{=}} \mathbb{E}(e^{itXY}|X=1)\mathbb{P}(X=1) + \mathbb{E}(e^{itXY}|X=-1)\mathbb{P}(X=-1) = \frac{1}{2}\underbrace{\mathbb{E}(e^{itY})}_{\varphi_Y(t)} + \frac{1}{2}\underbrace{\mathbb{E}(e^{-itY})}_{\varphi_Y(-t)}$$
$$= \frac{1}{2}\int_0^\infty e^{itx}e^{-x}dx + \frac{1}{2}\varphi_Y(-t) = \frac{1}{2(it-1)}\left[e^{x(it-1)}\right]_0^\infty + \frac{1}{2}\varphi_Y(-t) = \frac{1}{2(1-it)} + \frac{1}{2(1+it)}$$
$$= \frac{1}{1+t^2} = \frac{1}{1-it}\frac{1}{1+it} = \varphi_{Z_1-Z_2}(t).$$

4. Show by an example that $\phi_{X+Y}(u) = \phi_X(u)\phi_Y(u)$ does not necessarily imply that the random variables X and Y are independent. *Hint:* Think of a famous distribution!

   Solution: (by Dani, approved by Balázs)

   Let X be standard Cauchy distributed random variable. It is known that $\phi_X(u) = e^{-|u|}$. Thus, with $Y := X$ we get
$$\phi_{X+Y}(u) = \phi_{2X}(u) = \phi_X(2u) = e^{-|2u|} = e^{-|u|}e^{-|u|} = \phi_X(u)\phi_Y(u),$$
   though, X and Y are obviously not independent.

5. Let U, X and Y be independent random variables distributed as follows: $U \sim \text{UNI}[0,1]$, $X, Y \sim \text{EXP}(1)$. Use the method of characteristic functions to prove that

$$Z := U \cdot (X+Y) \sim \text{EXP}(1).$$

Solution (by Anonymous, approved by Balázs): $\varphi_X(t) = \varphi_Y(t) = \frac{1}{1-it}$ and X and Y are independent, thus $\varphi_{X+Y}(t) = \left(\frac{1}{1-it}\right)^2$. f is the p.d.f. of U, $(f(u) = \mathbb{1}_{[0,1]}(u))$.

$$\varphi_Z(t) = \mathbb{E}(e^{itU(X+Y)}) \overset{(*)}{=} \int_{-\infty}^{\infty} \mathbb{E}(e^{itu(X+Y)} \,|\, U=u) f(u)\,\mathrm{d}u \overset{(**)}{=} \int_{-\infty}^{\infty} \mathbb{E}(e^{itu(X+Y)}) f(u)\,\mathrm{d}u =$$
$$\int_{-\infty}^{\infty} \varphi_{X+Y}(ut) f(u)\,\mathrm{d}u = \int_0^1 \left(\frac{1}{1-itu}\right)^2 \mathrm{d}u =$$
$$\frac{-1}{it}\int_0^1 -it\,(1-itu)^{-2} = \frac{-1}{it}\left[\frac{(1-itu)^{-1}}{-1}\right]_{u=0}^1 = \frac{1}{1-it}, \quad (3)$$

where in $(*)$ we used the tower rule and in $(**)$ we used the independence of U and $X+Y$. Now $\frac{1}{1-it}$ is the characteristic function of an EXP(1) random variable, thus Z has EXP(1) distribution.

6. *The Lévy distribution is stable.* Let X denote a random variable with standard Lévy distribution. On the one hand, we have already learnt that $S_n/n^2 \Rightarrow X$, where $S_n = \eta_1 + \cdots + \eta_n$, where $\eta_1, \eta_2, \ldots$ are i.i.d. and η_k has the same distribution as the hitting time of level one by a one dimensional simple symmetric random walk starting from the origin. On the other hand, we have learnt that $\mathbb{E}(e^{itX}) = e^{-\sqrt{-2it}}$. Denote by LEVY$(a)$ the distribution of aX, where $a \in \mathbb{R}_+$.

Give two different proofs of the fact that for any $a, b \in \mathbb{R}_+$ we have

$$\text{LEVY}(a) * \text{LEVY}(b) \sim \text{LEVY}((\sqrt{a} + \sqrt{b})^2). \quad (4)$$

(The $*$ symbol denotes convolution)

Solution (by Laci, simplified a bit by Balázs):

Solution 1: Characteristic Function method:

We will show that $\varphi_{Levy(a)*Levy(b)} = \varphi_{Levy((\sqrt{a}+\sqrt{b})^2)}$

By page 89 of the lecture notes:

$$\varphi_{Levy(a)*Levy(b)} = \varphi_{Levy(a)} \cdot \varphi_{Levy(b)} = e^{-\sqrt{-2ita}-\sqrt{-2itb}} = e^{-\sqrt{-2it}(\sqrt{a}+\sqrt{b})}$$
$$= e^{-\sqrt{-2it(\sqrt{a}+\sqrt{b})^2}} = \varphi_{Levy((\sqrt{a}+\sqrt{b})^2)}$$

Solution 2: $S_n/n^2 \Rightarrow X$, so for any $\alpha \in \mathbb{R}_+$ we have $S_{\lfloor \alpha n \rfloor}/\lfloor \alpha n \rfloor^2 \Rightarrow X$. This and Slutsky imply

$$S_{\lfloor \alpha n \rfloor}/n^2 \Rightarrow \alpha^2 X \sim \text{LEVY}(\alpha^2). \quad (5)$$

Note that then $S_{n+m} - S_n$ has the same distribution as S_m, thus, analogously to (5), we obtain

$$\frac{S_{\lfloor (\alpha+\beta)n \rfloor} - S_{\lfloor \alpha n \rfloor}}{n^2} \Rightarrow \beta^2 X' \sim \text{LEVY}(\beta^2). \quad (6)$$

Also note that $S_{n+m} - S_n$ is independent of S_n, moreover $(S_{n+m} - S_n) + S_n = S_{n+m}$, thus

$$\frac{S_{\lfloor (\alpha+\beta)n \rfloor}}{n^2} = \frac{S_{\lfloor (\alpha+\beta)n \rfloor} - S_{\lfloor \alpha n \rfloor}}{n^2} + \frac{S_{\lfloor \alpha n \rfloor}}{n^2} \Rightarrow \alpha^2 X + \beta^2 X' \sim \text{LEVY}(\alpha^2) * \text{LEVY}(\beta^2). \quad (7)$$

On the other hand, analogously to (5), we have

$$\frac{S_{\lfloor (\alpha+\beta)n \rfloor}}{n^2} \Rightarrow (\alpha+\beta)^2 X \sim \text{LEVY}((\alpha+\beta)^2) \quad (8)$$

Thus, putting together (7) and (8), we obtain LEVY$(\alpha^2)*$LEVY$(\beta^2) \sim$ LEVY$((\alpha+\beta)^2)$. Taking $\alpha = \sqrt{a}$ and $\beta = \sqrt{b}$, we obtain the desired (4).

7. Let $X_1, X_2, X_3, \dots$ denote i.i.d. r.v.'s with UNI[0, 1] distribution. Use Lindeberg to show that

$$\frac{\sum_{k=1}^{n} kX_k - \frac{n^2}{4}}{\frac{1}{6}n^{\frac{3}{2}}} \Rightarrow N(0,1)$$

Solution (by Marci, approved by Balázs):

Let $Y_k := kX_k$. Then $Y_k \sim$ UNI[0, k], therefore $\mathbb{E}Y_k = \frac{k}{2}$ and $\mathbf{Var}Y_k = \frac{k^2}{12}$. Let us also denote $S_n := \sum_{k=1}^{n} kX_k = \sum_{k=1}^{n} Y_k$. Then $\mathbb{E}S_n = \frac{n^2+n}{4}$ and $\mathbf{Var}S_n = \frac{1}{12} \cdot \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$, since Y_k's are independent.

First let us use Lindeberg's theorem for the random variables $\xi_{n,k} = Y_k$, $k = 1, 2, \dots, n$. We have to check Lindeberg's condition. Using the notation of the theorem we have $\sigma_n^2 = \mathbf{Var}S_n = \frac{n^3}{36} + \mathcal{O}(n^2)$ and $|\tilde{\xi}_{n,k}| = |Y_k - \frac{k}{2}| \leq \frac{k}{2} \leq \frac{n}{2}$ for every $k = 1, 2, \dots, n$. Hence for a fixed $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathbb{E}\Big[|\tilde{\xi}_{n,k}|^2 \cdot \mathbb{1}\left[|\tilde{\xi}_{n,k}| > \varepsilon\sigma_n\right]\Big] \leq \lim_{n\to\infty} \frac{36}{n^3} \cdot n \cdot \mathbb{E}\left[\frac{n^2}{4} \cdot \mathbb{1}\left[\frac{n}{2} > \varepsilon\sigma_n\right]\right] =$$

$$= 9 \lim_{n\to\infty} \mathbb{E}\Big[\mathbb{1}\left[\frac{n}{2} > \varepsilon\sigma_n\right]\Big] = 9 \cdot \lim_{n\to\infty} \mathbb{1}\left[\frac{n}{2} > \varepsilon\sigma_n\right] \overset{(*)}{=} 0,$$

where at $(*)$ we used that $\sigma_n = \frac{n^{3/2}}{6} + \mathcal{O}(n)$, therefore there exists n_0 such that the condition in the indicator fails for every $n \geq n_0$.

Thus by Lindeberg's theorem we get

$$\frac{\sum_{k=1}^{n} kX_k - \frac{n^2}{4} - \frac{n}{4}}{\sigma_n} \Rightarrow N(0,1).$$

Hence we can conclude

$$\frac{\sum_{k=1}^{n} kX_k - \frac{n^2}{4}}{\frac{1}{6}n^{\frac{3}{2}}} \Rightarrow N(0,1)$$

by Slutsky.

8. For any $s \in (1, +\infty)$ let X_s denote an $\mathbb{N}_+$-valued random variable satisfying $\mathbb{P}(X_s = n) = n^{-s}/\zeta(s)$, where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Denote by Y_s the number of distinct primes that divide X_s. Show that

$$\frac{Y_{1+\varepsilon} - \ln(1/\varepsilon)}{\sqrt{\ln(1/\varepsilon)}} \Rightarrow \mathcal{N}(0,1), \qquad \varepsilon \to 0_+ \tag{9}$$

Hint: To approximate $\sum_{p\in\mathcal{P}} p^{-s}$, take the log of the Euler formula for the Riemann zeta function $\zeta(s)$.

Solution (by Dani, streamlined by Balázs):

Throughout the proof we will use the notation $s = 1 + \varepsilon$.

Let $Z_{p,s}$ be the indicator of the event that p divides X_s. It is easy to see that

$$Y_s = \sum_{p\in\mathcal{P}} Z_{p,s}.$$

From page 128 of the lecture notes we know that the indicator variables $(Z_{p,s})_{p\in\mathcal{P}}$ are independent, moreover

$$\mathbb{E}\left(Z_{p,s}\right) = \mathbb{P}\left(Z_{p,s} = 1\right) = p^{-s}. \tag{10}$$

Thus

$$\mathbb{E}\left(Y_s\right) = \sum_{p\in\mathcal{P}} \mathbb{E}\left(Z_{p,s}\right) = \sum_{p\in\mathcal{P}} p^{-s}, \quad \mathbb{D}^2\left(Y_s\right) = \sum_{p\in\mathcal{P}} \mathbb{D}^2\left(Z_{p,s}\right) = \sum_{p\in\mathcal{P}} p^{-s}(1-p^{-s}) = \sum_{p\in\mathcal{P}} p^{-s} - \sum_{p\in\mathcal{P}} p^{-2s} \tag{11}$$

Our first goal is to show

$$\mathbb{E}\left(Y_{1+\varepsilon}\right)=\log\left(\frac{1}{\varepsilon}\right)+O(1), \qquad \varepsilon\to 0. \tag{12}$$

We will prove this by taking the log of both sides of the Euler product formula for the Riemann zeta function. We begin by showing

$$\log\zeta(1+\varepsilon)=\log\left(\frac{1}{\varepsilon}\right)+O(1), \qquad \varepsilon\to 0_+ \tag{13}$$

using classical bounds obtained from the monotonicity of x^{-s}.

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}\geq\int_1^{\infty}x^{-s}dx=\left[-\frac{x^{-\varepsilon}}{\varepsilon}\right]_1^{\infty}=\frac{1}{\varepsilon}\Rightarrow\log\zeta(1+\varepsilon)\geq\log\left(\frac{1}{\varepsilon}\right)$$
$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=1+\sum_{n=2}^{\infty}n^{-s}\leq 1+\int_1^{\infty}x^{-s}dx=1+\frac{1}{\varepsilon}\Rightarrow\log\zeta(1+\varepsilon)\leq\log\left(1+\frac{1}{\varepsilon}\right)$$

The two bounds differ by $O(1)$, since $0<\log\left(1+\frac{1}{\varepsilon}\right)-\log\left(\frac{1}{\varepsilon}\right)=\log\left(1+\varepsilon\right)\leq\log 2$.
Another technical observation is

$$\sum_{p\in\mathcal{P}}p^{-2s}=O(1), \tag{14}$$

which comes from $0<\sum_{p\in\mathcal{P}}p^{-2s}\leq\sum_{p\in\mathcal{P}}p^{-2}\leq\sum_{n=1}^{\infty}n^{-2}<+\infty$.
With all in this in mind, we will show (12). Notice that taking the log of both sides of Euler's product formula for the Riemann zeta function (see page 128) we get

$$\log\zeta(s)=\log\left(\prod_{p\in\mathcal{P}}\frac{1}{1-p^{-s}}\right)=-\sum_{p\in\mathcal{P}}\log\left(1-p^{-s}\right)\stackrel{(*)}{=}\sum_{p\in\mathcal{P}}(p^{-s}+O(p^{-2s}))\stackrel{(14)}{=}\sum_{p\in\mathcal{P}}p^{-s}+O(1),$$

where in $(*)$ we used the first order Taylor expansion $\log(1+x)=x+O(x^2)$. Putting this together with (13) we obtain (12). Next we observe

$$\mathbb{D}^2\left(Y_s\right)\stackrel{(11),(14)}{=}\sum_{p\in\mathcal{P}}p^{-s}+O(1)\stackrel{(12)}{=}\log\left(\frac{1}{\varepsilon}\right)+O(1). \tag{15}$$

This means that instead of proving (9), it is enough to prove

$$\frac{Y_{1+\varepsilon}-\mathbb{E}\left(Y_{1+\varepsilon}\right)}{\mathbb{D}\left(Y_{1+\varepsilon}\right)}\Rightarrow\mathcal{N}(0,1), \qquad \varepsilon\to 0, \tag{16}$$

because $\mathbb{D}\left(Y_{1+\varepsilon}\right)/\sqrt{\log\left(\frac{1}{\varepsilon}\right)}\to 1$ and $\left(\mathbb{E}\left(Y_{1+\varepsilon}\right)-\log\left(\frac{1}{\varepsilon}\right)\right)/\sqrt{\log\left(\frac{1}{\varepsilon}\right)}\to 0$ as $\varepsilon\to 0$, so (9) and (16) are indeed equivalent by Slutsky.

To show (16), we only need to check Lindeberg's condition. We make two technical observations before that. Firstly, $Y_{1+\varepsilon}$ is formally an infinite sum, but of course only finitely many terms are non-zero, since $\mathbb{P}(Y_s<+\infty)=1$ implies that Y_s only has finitely many distinct prime divisors. Thus it makes sense to use Lindeberg in this case even when $N_n=\infty$. Secondly, we do not have a triangular array where the rows are indexed by $n=1,2,\dots$ but they are indexed by a continuous parameter ε. But this is not a problem, because is we show that (16) holds along any sequence $\varepsilon_n\to 0$ as $n\to\infty$ using Lindeberg, then (16) also follows.

Since he symbol ε is already taken, we will fix a $\delta>0$ instead in Lindeberg's condition. Let us define the centered random variables $\tilde{Z}_{p,1+\varepsilon}=Z_{p,1+\varepsilon}-\mathbb{E}(Z_{p,1+\varepsilon})$. We need to check that

$$\frac{1}{\mathbb{D}^2\left(Y_{1+\varepsilon}\right)}\sum_{p\in\mathcal{P}}\mathbb{E}\left(\left|\tilde{Z}_{p,1+\varepsilon}\right|^2\chi_{\left\{\left|\tilde{Z}_{p,1+\varepsilon}\right|>\delta\mathbb{D}\left(Y_{1+\varepsilon}\right)\right\}}\right)\to 0, \qquad \varepsilon\to 0.$$

Now observe that $|Z_{p,1+\varepsilon}|$ is bounded by 1 yet for all fixed $\delta>0$ we have $\delta\mathbb{D}\left(Y_{1+\varepsilon}\right)\to\infty$ as $\varepsilon\to 0_+$, hence all the terms inside the expectation become simultaneously 0 for small enough ε. Thus Lindeberg's condition holds and by Lindeberg's theorem (16) follows.

9. Prove that X_n converges to 0 in probability if and only if $\varphi_n(t) \to 1$ in an open neighbourhood of $t = 0$.

Solution (by Laci and Dani, approved by Balázs):

$\Rightarrow$: (by Laci) If X_n converges to 0 in probability then $X_n \Rightarrow 0$, thus $\varphi_n(t) = \mathbb{E}(e^{itX}) \to \mathbb{E}(e^{it0}) = 1$ by the second theorem on page 91 of the scanned lecture notes.

$\Leftarrow$: (by Dani)

Let $[-\delta, \delta]$ be an interval in which $\varphi_n(t) \to 1$ as $n \to \infty$.

Let us recall a calculation from the proof of Lévy's continuity theorem (see page 109):

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi_n(t)dt = \frac{1}{2\delta}\int_{-\delta}^{\delta} \mathbb{E}\left(e^{itX_n}\right)dt = \mathbb{E}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{itX_n}dt\right) =$$
$$\mathbb{E}\left(\frac{e^{i\delta X_n} - e^{-i\delta X_n}}{2i\delta X_n}\right) = \mathbb{E}\left(\frac{sh\left(i\delta X_n\right)}{i\delta X_n}\right) = \mathbb{E}\left(\frac{\sin\left(\delta X_n\right)}{\delta X_n}\right)$$

Idea: the l.h.s. obviously converges to 1 as $n \to \infty$ by the dominated convergence theorem. On the r.h.s. though we have the expectation of function of the form $\frac{\sin x}{x}$ where it is defined as 1 at the $x = 0$ case. We know that the unique maximum of $x \mapsto \frac{\sin x}{x}$ is attained at $x = 0$ with value 1. This means all of the mass should concentrate at $x = 0$, otherwise, the expectation would be less than 1. This means $X_n \Rightarrow 0$.

The precise version of this observation is the following:

$$\mathbb{E}\left(\frac{\sin\left(\delta X_n\right)}{\delta X_n}\right) = \mathbb{E}\left(\frac{\sin\left(\delta X_n\right)}{\delta X_n}\mathbb{1}_{[|X_n|\geq\varepsilon]}\right) + \mathbb{E}\left(\frac{\sin\left(\delta X_n\right)}{\delta X_n}\mathbb{1}_{[|X_n|<\varepsilon]}\right) \leq$$
$$\frac{sin(\delta\varepsilon)}{\delta\varepsilon}\mathbb{P}\left(|X_n| \geq \varepsilon\right) + \mathbb{P}\left(|X_n| < \varepsilon\right) = \left(\frac{sin(\delta\varepsilon)}{\delta\varepsilon} - 1\right)\mathbb{P}\left(|X_n| \geq \varepsilon\right) + 1$$

Since $\frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi_n(t)dt = \frac{1}{2\delta}\int_{-\delta}^{\delta} \mathbb{E}\left(e^{itX_n}\right)dt \to 1$ as $n \to \infty$ for all $\rho > 0$ we can choose an N such that $n \geq N$ imply $\frac{1}{2\delta}\int_{-\delta}^{\delta} \varphi_n(t)dt = \frac{1}{2\delta}\int_{-\delta}^{\delta} \mathbb{E}\left(e^{itX_n}\right)dt \geq 1 - \rho$, thus

$$1 - \rho \leq \left(\frac{sin(\delta\varepsilon)}{\delta\varepsilon} - 1\right)\mathbb{P}\left(|X_n| \geq \varepsilon\right) + 1$$
$$\mathbb{P}\left(|X_n| \geq \varepsilon\right) \leq \frac{\rho}{1 - \frac{sin(\delta\varepsilon)}{\delta\varepsilon}} =: \rho'$$

ρ' can be set arbitrarily small for a given ε by setting ρ small enough. Hence, $X_n \Rightarrow 0$.

10. Let $X_1, X_2, \ldots$ be i.i.d. random variables. Assume $\mathbb{P}(X_i \geq 0) = 1$, $\mathbb{E}X_i = 1$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Prove that
$$2\left(\sqrt{S_n} - \sqrt{n}\right) \Rightarrow \mathcal{N}(0, \sigma^2).$$

Solution (by Dani, approved by Balázs):

$$\mathcal{N}(0,\sigma^2) \Leftarrow \frac{S_n - n}{\sqrt{n}} = 2\left(\sqrt{S_n} - \sqrt{n}\right)\frac{\sqrt{S_n} + \sqrt{n}}{2\sqrt{n}} = 2\left(\sqrt{S_n} - \sqrt{n}\right)\underbrace{\frac{\sqrt{\frac{S_n}{n}} + 1}{2}}_{\Rightarrow 1}$$

The last limit comes from the law of large numbers. The rest of the proof is a simple application of Slutsky's theorem.

11. For each $n \in \mathbb{N}$, let $\xi_{n,k}, k = 1, \ldots, n$ denote i.i.d. random variables with BER$(1/n)$ distribution. These random variables form a triangular array. Let $S_n = \xi_{n,1} + \cdots + \xi_{n,n}$. Find the weak limit of
$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\text{Var}(S_n)}}, \qquad n \to \infty.$$

Explain why this is a valuable lesson in the context of Lindeberg's theorem.

Solution (By Dani, expanded a bit by Balázs): $S_n \sim \mathrm{BIN}(n, 1/n)$, so $\mathbb{E}(S_n) = 1$ and $\mathbb{D}^2(S_n) = 1 - \frac{1}{n}$, which goes to 1 as $n \to \infty$. We know from page 91-92 of the scanned lecture notes that $S_n \implies Y$, where $Y \sim \mathrm{POI}(1)$. Thus by Slutsky we obtain

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \implies Y - 1, \qquad n \to \infty.$$

Obviously, the limiting distribution is not standard normal, so something must go wrong with Lindeberg's condition. If we denote $\sigma_{n,k}^2 = \mathrm{Var}(\xi_{n,k}) = \frac{1}{n}(1 - \frac{1}{n})$ and $\sigma_n^2 = \mathrm{Var}(S_n)$ (this is the standard Lindeberg notation used on page 116), then we see that in our case

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{\sigma_{n,k}^2}{\sigma_n^2} = \lim_{n\to\infty} \frac{1}{n} = 0, \tag{17}$$

which looks promising (c.f. equation A from page 117), however (17) is weaker than Lindeberg's condition! To see how the Lindeberg condition fails, let $\tilde{\xi}_{n,k} = \xi_{n,k} - \mathbb{E}(\xi_{n,k}) = \xi_{n,k} - 1/n$, and let us take a look at

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathbb{E}\left(|\tilde{\xi}_{n,k}|^2 \chi_{\{|\tilde{\xi}_{n,k}| > \varepsilon\sigma_n\}} \right)$$

Note that by choosing $\varepsilon < \frac{1}{2}$ we can get $\varepsilon\sigma_n = \varepsilon\sqrt{1 - \frac{1}{n}} < \varepsilon < \frac{1}{2}$. Thus, $|\tilde{\xi}_{n,k}| > \varepsilon\sigma_n$ implies $\xi_{n,k} = 1$ and $\tilde{\xi}_{n,k} = 1 - \frac{1}{n}$ for $n \ge 3$. Hence,

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{n} \mathbb{E}\left(|\tilde{\xi}_{n,k}|^2 \chi_{\{|\tilde{\xi}_{n,k}| > \varepsilon\sigma_n\}} \right) = \frac{1}{1 - \frac{1}{n}} \sum_{k=1}^{n} \left(1 - \frac{1}{n}\right)^2 \mathbb{P}\left(\xi_{n,k} = 1\right) = 1 - \frac{1}{n} \to 1 \ne 0.$$

Thus Lindeberg's condition fails for this exercise.