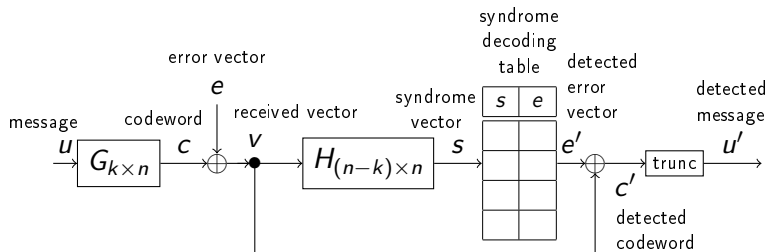# 3. Binary Linear Codes

Coding Technology

For linear codes,

$$c = uG,$$

then from the received vector v, we compute the syndrome

$$s^T = Hv^T$$

and detect the error based on s.

For systematic codes, we can obtain u' from c' by truncation.

Consider the following generator matrix G and parity check matrix H:

$$G_{2\times 5} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad H_{3\times 5} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $GH^T = 0$ holds.

G and H together define a $C(5, 2)$ code.

What are the code vectors of this code?

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Solution.

$$c^{(0)} = u^{(0)} G = (00) \cdot \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = (00000)$$

$$c^{(1)} = u^{(1)} G = (01) \cdot \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = (01111)$$

$$c^{(2)} = u^{(2)} G = (10) \cdot \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = (10110)$$

$$c^{(3)} = u^{(3)} G = (11) \cdot \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = (11001)$$

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Solution. d_{min}, the minimal Hamming-distance between codewords can be computed either by pairwise comparison of codewords, or, for linear codes, the following also holds:

$$d_{min} = \min_{c \neq (00...0)} w(c).$$

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Solution. $d_{\min}$, the minimal Hamming-distance between codewords can be computed either by pairwise comparison of codewords, or, for linear codes, the following also holds:

$$d_{\min} = \min_{c \neq (00...0)} w(c).$$

Using either method, we get $d_{\min} = 3$, so the error detection capability of this code is

$$d_{\min} - 1 = 2,$$

and the code correction capability is

$$\lfloor (d_{\min} - 1)/2 \rfloor = 1.$$

For systematic codes, the leftmost square block of G is the identity matrix:

$$G = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array} \right].$$

Then, using

$$G = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array} \right],$$
$$\underbrace{}_{B}$$

H can be computed as

$$H = (B^T, I_{n-k}) = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right].$$
$$\underbrace{}_{B^T}$$

For decoding, we first compute the syndrome vector from the output using the parity check matrix H:

$$s^T = Hv^T.$$

Note that

$$s^T = Hv^T = He^T$$

always holds. In other words, the syndrome only depends on the error vector, not the original codeword.

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Next step: based on the syndrome vector s, we try to guess the error vector e.

The syndrome vector s is shorter than the error vector $e \rightarrow$ there are multiple possible error vectors for each syndrome.

For an s syndrome vector, its error group is

$$E_s = \{e : He^T = s^T\}.$$

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Approach 1 (naive approach):

1. List all possible error vectors (binary vectors of length n).
2. Compute the multiplications $He^T = s^T$, $\forall e \in \{0, 1\}^n$.
3. Group the results according to s.

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3. Group the results according to s.

Approach 2: for any error e, its error group consists of the vectors

$$\left\{ e, e + c^{(1)}, \ldots, e + c^{(2^k - 1)} \right\},$$

where $c^{(i)}$ are the codewords.

Compute the error group of $e = (0\,1\,1\,1\,0)$ for the previous code. Also compute the syndrome vector.

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Solution. The syndrome vector is

$$s^T = He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The error group is

$$\begin{aligned} E_{(001)} =& \{(01110), (01110) + (01111), \\ & (01110) + (10110), (01110) + (11001)\} = \\ & \{(01110), (00001), (11000), (10111)\}. \end{aligned}$$

Next we decide which error to choose from the error group during decoding. This error vector will be called the group leader.

Assuming e.g. $P_b = 0.01$ (bit error probability), we have

$$E_{001} = \{ \quad (01110), \quad (00001), \quad (11000), \quad (10111) \quad \}$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$w = 3 \qquad w = 1 \qquad w = 2 \qquad w = 4$$

$$9.8 \cdot 10^{-7} \quad 9.6 \cdot 10^{-3} \quad 9.7 \cdot 10^{-5} \quad 9.9 \cdot 10^{-9}$$

The group leader is $e = (00001)$, as it has the largest probability (and, equivalently, smallest weight).

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The group leader is $e = (00001)$, as it has the largest probability (and, equivalently, smallest weight).

In general, from each error group, the group leader will be the vector with minimal weight.

The syndrome decoding table is constructed next. Approach 1:

1. List all possible error vectors (all binary vectors of length n).
2. Compute the products $He^T = s^T$, $\forall e \in \{0, 1\}^n$.
3. Group the results according to the value of s.

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Approach 2:

1. Pick an error vector e and compute the corresponding syndrome vector s: $He^T = s^T$.
2. $E_s = \left\{ e, e + c^{(1)}, \ldots, e + c^{(2^k - 1)} \right\}$.
3. Pick an error vector e not listed in any of the groups yet and repeat.

Constructing the syndrome decoding table

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1. Pick an error vector e and compute the corresponding syndrome vector s: $He^T = s^T$.
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For both approaches:

4. Determine the vector e with minimum weight in each group.
5. For each s and corresponding group leader e, the pair (s, e) goes into the syndrome decoding table.

We will use approach 2.

1. Pick $e = (00000)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(000)} = & \{(00000), (00000) + (01111), \\ & (00000) + (10110), (00000) + (11001)\} = \\ & \{(00000), (01111), (10110), (11001)\}. \end{aligned}$$

Example (continued).

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Example (continued).

1. Pick $e = (00010)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(010)} = & \{(00010), (00010) + (01111), \\ & (00010) + (10110), (00010) + (11001)\} = \\ & \{(00010), (01101), (10100), (11011)\}. \end{aligned}$$

Example (continued).

1. Pick $e = (00100)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(100)} = & \{(00100), (00100) + (01111), \\ & (00100) + (10110), (00100) + (11001)\} = \\ & \{(00100), (01011), (10010), (11101)\}. \end{aligned}$$

Example (continued).

1. Pick $e = (01000)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(111)} = \{&(01000), (01000) + (01111), \\ &(01000) + (10110), (01000) + (11001)\} = \\ &\{(01000), (00111), (11110), (10001)\}. \end{aligned}$$

Example (continued).

1. Pick $e = (10000)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(110)} = & \{(10000), (10000) + (01111), \\ & (10000) + (10110), (10000) + (11001)\} = \\ & \{(10000), (11111), (00110), (01001)\}. \end{aligned}$$

Example (continued).

1. Pick $e = (00101)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(101)} = & \{(00101), (00101) + (01111), \\ & (00101) + (10110), (00101) + (11001)\} = \\ & \{(00101), (01010), (10011), (11100)\}. \end{aligned}$$

Example (continued).

1. Pick $e = (00011)$.

$$He^T = \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

2.

$$\begin{aligned} E_{(011)} = &\{(00011), (00011) + (01111), \\ &(00011) + (10110), (00011) + (11001)\} = \\ &\{(00011), (01100), (10101), (11010)\}. \end{aligned}$$

Constructing the syndrome decoding table

From each group, the error vector with minimal weight is the group leader.

$E_{(000)} = \{(00000), (01111), (10110), (11001)\} \rightarrow e_{(000)} = (00000)$

$E_{(001)} = \{(00001), (01110), (10111), (11000)\} \rightarrow e_{(001)} = (00001)$

$E_{(010)} = \{(00010), (01101), (10100), (11011)\} \rightarrow e_{(010)} = (00010)$

$E_{(011)} = \{(00011), (01100), (10101), (11010)\} \rightarrow e_{(011)} = (00011)$

$E_{(100)} = \{(00100), (01011), (10010), (11101)\} \rightarrow e_{(100)} = (00100)$

$E_{(101)} = \{(00101), (01010), (10011), (11100)\} \rightarrow e_{(101)} = (00101)$

$E_{(110)} = \{(00110), (01001), (10000), (11111)\} \rightarrow e_{(110)} = (10000)$

$E_{(111)} = \{(00111), (01000), (10001), (11110)\} \rightarrow e_{(111)} = (01000)$

The syndrome decoding table

syndrome vector	group leader
(000)	(00000)
(001)	(00001)
(010)	(00010)
(011)	(00011)
(100)	(00100)
(101)	(00101)
(110)	(10000)
(111)	(01000)

The standard array

The standard array lists the errors in each error group according to increasing weight.

syndrome vector	1	2	3	4
(000)	(00000)	(10110)	(11001)	(01111)
(001)	(00001)	(11000)	(01110)	(10111)
(010)	(00010)	(10100)	(01101)	(11011)
(011)	(00011)	(01100)	(10101)	(11010)
(100)	(00100)	(10010)	(01011)	(11101)
(101)	(00101)	(01010)	(10011)	(11100)
(110)	(10000)	(01001)	(00110)	(11111)
(111)	(01000)	(10001)	(00111)	(11110)

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syndrome vector	1	2	3	4
(000)	(00000)	(10110)	(11001)	(01111)
(001)	(00001)	(11000)	(01110)	(10111)
(010)	(00010)	(10100)	(01101)	(11011)
(011)	(00011)	(01100)	(10101)	(11010)
(100)	(00100)	(10010)	(01011)	(11101)
(101)	(00101)	(01010)	(10011)	(11100)
(110)	(10000)	(01001)	(00110)	(11111)
(111)	(01000)	(10001)	(00111)	(11110)

The code can correct t errors $\iff$ in each group with minimal weight t or less, the group leader is unique.
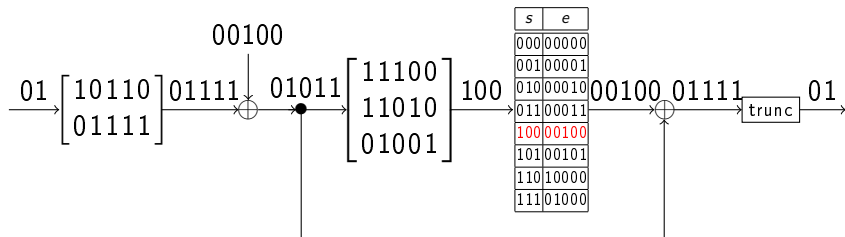
Compute the steps of this coding scheme for message vector (01)
and error vector (00100).

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Solution.

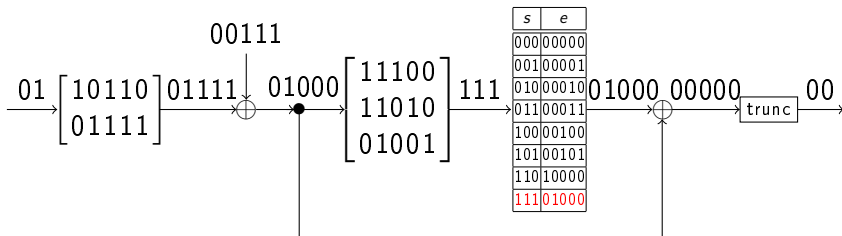Compute the steps of this coding scheme for message vector (01) and error vector (00111).

Compute the steps of this coding scheme for message vector (01) and error vector (00111).

Solution.

s	e
000	00000
001	00001
010	00010
011	00011
100	00100
101	00101
110	10000
111	01000

$$01 \begin{bmatrix} 10110 \\ 01111 \end{bmatrix} 01111 \xrightarrow{00111} 01000 \begin{bmatrix} 11100 \\ 11010 \\ 01001 \end{bmatrix} 111 \quad 01000 \quad 00000 \xrightarrow{\text{trunc}} 00$$

Compute the steps of this coding scheme for message vector (01) and error vector (00111).

Solution.
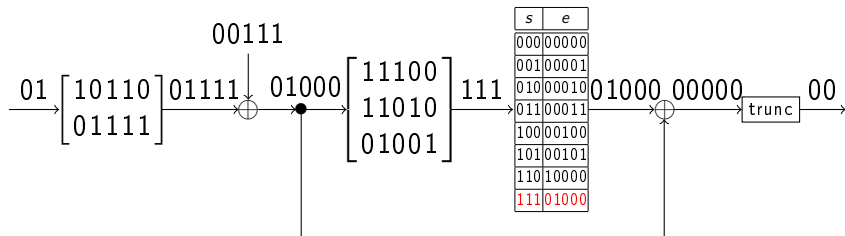
The error vector (00111) has a small probability $\rightarrow$ no problem if this happens (communication QoS).

Errors which are one of the group leaders can be corrected:

$$(00000) \quad (00001) \quad (00010) \quad (00011)$$
$$(00100) \quad (00101) \quad (10000) \quad (01000)$$

These include all error vectors with weight 0 and 1, and 2 error vectors with weight 2. All other error vectors cannot be corrected, so the block error probability is

$$P_e = \left(\binom{5}{2} - 2 \right) P_b^2 (1 - P_b)^3 + \binom{5}{3} P_b^3 (1 - P_b)^2 + 5 P_b^4 (1 - P_b) + P_b^5;$$

for $P_b = 0.1$,

$$P_e \approx 0.0669.$$

What do we gain by using an error correcting code?

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Consider what happens when we transmit messages with no coding on the same channel. For a 2-bit block, the probability of erroneous decoding is

$$1 - (1 - P_b)^2 = 0.19,$$

while it is 0.0669 when using error correcting code. (On the other hand, the code rate is 2/5, so using the code effectively reduces the channel capacity to 2/5 of the original capacity. Tradeoff.)

Another possible comparison is to compare to another channel with different bit error probability P'_b where messages of block length 5 are transmitted without any coding. Compute P'_b so that the block error probability is the same as for the original channel with coding.

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For the original channel, block error probability is 0.0669; for the channel with no coding, it is

$$1 - (1 - P'_b)^5 = P_e = 0.0669 \quad \rightarrow \quad P'_b \approx 0.0137.$$

By using the error correcting code, we can obtain the same block error probability over a noisier channel ($P_b = 0.1 > 0.0137 = P'_b$).

A binary systematic code is given by its codewords:

$$c^{(0)} = (000000), \quad c^{(1)} = (010101),$$
$$c^{(2)} = (101010), \quad c^{(3)} = (111111).$$

(a) What is the type of the code?

(b) Compute the error detection and error correction capabilities of this code.

(c) Compute the generator and parity check matrices.

(d) Compute the syndrome and the error group of $e = (010100)$. What is the group leader? Do we decode correctly for $e = (010100)$?

(e) Compute the syndrome decoding table.

$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$

(a) The type of the code is C(6,2) as the length of the codewords is $n = 6$ and the number of codewords is $4 = 2^k$, so $k = 2$.

$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$

(a) The type of the code is C(6,2) as the length of the codewords is $n = 6$ and the number of codewords is $4 = 2^k$, so $k = 2$.

(b) For linear codes,

$$d_{min} = \min_{c \neq (00...0)} w(c),$$

so the error detection capability of this code is $3 - 1 = 2$, and the code correction capability is

$$\left\lfloor \frac{d_{min} - 1}{2} \right\rfloor = 1,$$

that is, the code can correct 1 error.

$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$

(c) Row 1 of the generator matrix G is $c^{(2)}$, and row 2 is $c^{(1)}$:

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The code is systematic, since the leftmost 2×2 block of G is the identity matrix.

$$G = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix} \end{bmatrix}.$$

$c^{(0)} = (000000), c^{(1)} = (010101), c^{(2)} = (101010), c^{(3)} = (111111)$

(c)

$$G = \begin{bmatrix} 1 & 0 & \boxed{\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}} \\ 0 & 1 & \end{bmatrix}$$

$$B$$

For systematic codes, the parity check matrix H can be written as

$$H = (B^T, I_{n-k}) = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}.$$

$$B^T$$

The syndrome is $He^T = \begin{bmatrix} 101000 \\ 010100 \\ 100010 \\ 010001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$

$$\begin{aligned} E_{(0001)} =& \{(010100), (010100) + (010101), \\ & (010100) + (101010), (010100) + (111111)\} = \\ & \{(010100), \color{red}{(000001)}, (111110), (101011)\}. \end{aligned}$$

$e = (010100)$ is not the group leader, so the decoding will be erroneous.

Syndrome vector	group leader error vector
0001	000001
0010	000010
0100	000100
1000	001000
0101	010000
1010	100000
0011	000011
1001	001001
1011	100001
0110	000110
0111	010010
1111	110000
1100	001100
1110	100100
1101	011000

(e)

A systematic binary linear code has codewords

$$(0000), (0110), (1001), (1111).$$

(a) What are the parameters of the code?

(b) Compute the generator matrix G and the parity check matrix H.

(c) What are the error detecting and correcting capabilities of the code?

(d) Compute the syndrome and error group of $e = (0100)$.

Solution.

(a) $n = 4, k = 2$.

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(b) The rows of G are the codewords starting with 10 and 01, so

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

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$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(c) $d_{\min} = \min_{c \neq (00\ldots0)} w(c) = 2$, so the code can detect $d_{\min} - 1 = 1$ error and correct $\lfloor (d_{\min} - 1)/2 \rfloor = 0$ errors.

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$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

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(d) $s^T = He^T = \begin{bmatrix} 0110 \\ 1001 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

$E_{(10)} = \{(0100), (0100) + (0110), (0100) + (1001), (0100) + (1111)\} = \{(0100), (0010), (1101), (1011)\}$