2. Basic concepts of codes and the generic coding scheme

Coding Technology

Problem 1

A code has codewords

$$10100, 01111, 11110, 00000.$$

(a) Calculate the n and k parameters of the code.
(b) What is the minimal Hamming distance between codewords?
(c) How many errors can the code detect? How many errors can the code correct?

Problem 1

Solution.

(a) The length of the codewords is $n = 5$, and the number of codewords is $2^k = 4$ (one for each message vector of length k), so $k = 2$. This is a $C(5, 2)$ code.

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(b) Using pairwise comparison, the minimal Hamming distance is

$$d_{\min} = 2.$$

(c) A code with $d_{\min} = 2$ can detect

$$d_{\min} - 1 = 1$$

errors and correct

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = 0$$

errors.

Problem 2

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Solution.

(a) There are 32 binary vectors of length $n = 5$, and we have to choose $2^k = 2^2 = 4$ of them. We need to check their minimal pairwise Hamming distance, and choose 4 vectors where the minimal Hamming distance is as large as possible.

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4 codewords of length 5 with $d_{\min} \geq 4$ is not possible. (Even 3 codewords of length 5 with $d_{\min} \geq 4$ is not possible. Why?)

Problem 2

(b) If we assign the messages to the codewords according to the following list, then the lookup-table (LUT) is the same assignment in reverse:

$$\begin{aligned} 00 &\rightarrow 00000 \\ 01 &\rightarrow 11100 \\ 10 &\rightarrow 00111 \\ 11 &\rightarrow 11011 \end{aligned} \quad \rightarrow$$

c'	u'
00000	00
11100	01
00111	10
11011	11

The LUT above includes only the codewords, other received vectors are decoded to the codeword with minimal Hamming-distance.

(In case of $d = 2$, the received vector may have minimal Hamming-distance to multiple codewords. In such a situation, we may choose any of the codewords with minimal Hamming-distance for decoding.)

2. feladat

(b) The full $c' \to u'$ assignment for all possible received vectors is as follows:

c'	u'
00000, 00001, 00010, 00100, 01000, 10000, 01001, 10001, 10010	00
11100,11101,11110,11000,10100,01100, 01101, 01110	01
00111,00110, 00101, 00011, 01111, 10111, 10110, 10101	10
11011, 1110, 11001, 11111, 10011, 01011, 01010	11

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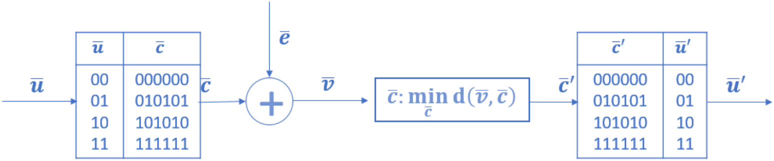
This table is significantly larger than the LUT, but it is not necessary to compute $\{c : \min d(c, c')\}$.

(c) Error detection: $d_{\text{min}} - 1 = 2$.
Error correction: $\lfloor \frac{d_{\text{min}} - 1}{2} \rfloor = 1$.

Problem 3

We have the following coding scheme:



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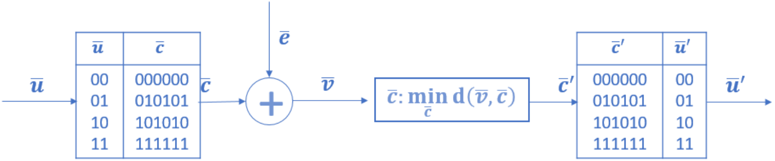


For $u = (11)$ and $e = (001000)$, determine the vectors c, v, c', u'.

Solution.

$$c = (111111)$$

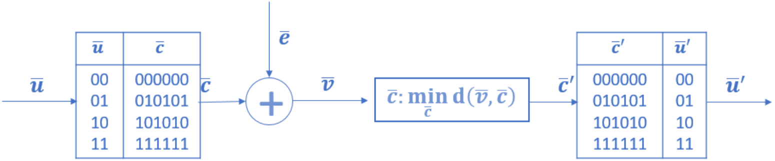
$$v = c + e = (111111) + (001000) = (110111),$$

$$c' = \{c : \min_c d(v, c)\} = (111111)$$

$$u' = (11)$$

Problem 4

Use the same coding scheme:



for $u = (01)$ and $e = (001011)$ to determine the vectors c, v, c', u'.

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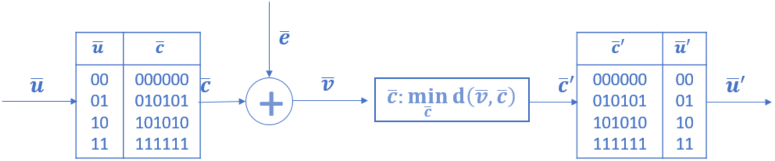

for $u = (01)$ and $e = (001011)$ to determine the vectors c, v, c', u'.

Solution.

$$
\begin{aligned}
c &= (010101) \\
v &= c + e = (010101) + (001011) = (011110), \\
c' &= \{c : \min_{c} d(v, c)\} = (111111) \\
u' &= (11)
\end{aligned}
$$

Problem 5

For each of the following sets of codewords, give the appropriate (n, k, d) designation, where n is number of bits in each codeword, k is the number of message bits transmitted by each codeword and $d = d_{\text{min}}$ is the minimum Hamming distance between codewords. Also give the code rate.

(a) $\{111, 100, 010, 001\}$

(b) $\{00000, 01111, 10100, 11011\}$

(c) $\{00000\}$

Problem 5

Solution.

(a) $\{111, 100, 010, 001\}$

- $n = 3$ (the length of the codewords);
- $k = 2$ (the number of codewords is $4 = 2^k$);
- $d = d_{\min} = 2$ (from pairwise comparison).
- the code rate is $k/n = 2/3$.

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$n = 5, k = 2, d = 2$, the code rate is $2/5$.

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(b) $\{00000, 01111, 10100, 11011\}$

$n = 5, k = 2, d = 2$, the code rate is $2/5$.

(c) $\{00000\}$

A bit of a trick question. $n = 5, k = 0$, d is undefined. The code rate is 0.

With only one codeword, no useful information can be transmitted since the receiver knows in advance what the codeword is going to be.

Problem 6

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Solution. We need a $C(5, 2, 3)$ block code. We have seen one like that in Problem 2, that code is suitable for this problem too: $\{(00000), (00111), (11100), (11011)\}$.

Problem 7

Pairwise Communications has developed a block code with three data bits (D_1, D_2, D_3) and three parity bits (P_1, P_2, P_3):

$$P_1 = D_1 + D_2, \quad P_2 = D_2 + D_3, \quad P_3 = D_3 + D_1.$$

(a) Calculate (n, k) for this code.

(b) What are the codewords?

(c) What is the minimum Hamming distance of the code?

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(b) The 8 possible codewords:

$$(000000), \quad (001011), \quad (010110), \quad (011101),$$
$$(100101), \quad (101110), \quad (110011), \quad (111000).$$

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(a) $n = 6, k = 3$.

(b) The 8 possible codewords:

$$\begin{array}{cccc} (000000), & (001011), & (010110), & (011101), \\ (100101), & (101110), & (110011), & (111000). \end{array}$$

(c) By pairwise inspection, the minimum Hamming distance is $d_{\min} = 3$.

Problem 8

The receiver computes three syndrome bits E_1, E_2 and E_3 from the (possibly corrupted) received data and parity bits:

$$E_1 = D_1 + D_2 + P_1, \quad E_2 = D_2 + D_3 + P_2, \quad E_3 = D_3 + D_1 + P_3.$$

The receiver performs maximum likelihood decoding using the syndrome bits. Determine the result of the maximum-likelihood decoder from among the following:

- no errors, or
- single error in a specific bit (state which one), or
- multiple errors,

for each of the following combination of syndrome bits:

$$E_1 E_2 E_3 = 000, \qquad E_1 E_2 E_3 = 010,$$
$$E_1 E_2 E_3 = 101, \qquad E_1 E_2 E_3 = 111.$$

Problem 8

Solution. Main points to consider:

- no errors result in 0 for all of E_1, E_2 and E_3.
- if there is only one error, and it is from among D_1, D_2 or D_3, then two of E_1, E_2 and E_3 will be 1's.
- if there is only one error, and it is from among P_1, P_2 or P_3, then one of E_1, E_2 and E_3 will be 1's.
- The ML decoder will pick the result with the fewest errors.

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For $E_1 E_2 E_3 = 000$, the result is no errors.

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For $E_1 E_2 E_3 = 010$, the result is 1 error in P_2.

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For $E_1 E_2 E_3 = 101$, the result is 1 error in D_1.

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For $E_1 E_2 E_3 = 101$, the result is 1 error in D_1.

For $E_1 E_2 E_3 = 111$, the result is multiple errors.

Problem 9

Is the following code an MDS (Maximum Distance Separable) code?

$$c_1 = (00000), \qquad c_2 = (11111)$$

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Solution. $n = 5$ $2^k = 2$, so $k = 1$, and $d_{\min} = 5$. A code is MDS if

$$d_{\min} = n - k + 1$$

holds. In this case,

$$d_{\min} = 5 = n - k + 1 = 5 - 1 + 1$$

holds, so this is an MDS.

Problem 10

Is the following code an MDS code?

$$\begin{array}{cccc} (000000), & (001011), & (010110), & (011101) \\ (100101), & (101110), & (110011), & (111000) \end{array}$$

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Solution. $n = 6, k = 3, d_{\min} = 3.$

$$n - k + 1 = 6 - 3 + 1 \neq d_{\min} = 3$$

No, it is not MDS.

Problem 11

A perfect code with $n = 15$ corrects $t = 1$ error. What is the value of k?

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Solution. For a perfect code, there is equality in the Hamming bound:

$$\sum_{i=0}^{t} \binom{n}{i} = 2^{n-k}$$

$$n + 1 = 2^{n-k}$$

$$16 = 2^{15-k},$$

so $k = 11$.